Cambridge Assessment International Education
Cambridge International Advanced Level

FURTHER MATHEMATICS | **9231/22**

Paper 2 | **October/November 2018**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2018 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Abbreviations

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Question	Answer	Marks	Guidance
1	$v_A^2 = \omega^2 (a^2 - 0.1^2)$ *and* $v_B^2 = \omega^2 (a^2 - 0.5^2)$	**B1**	Use $v^2 = \omega^2 (a^2 - x^2)$ at *A and B* (may be implied)
	$a^2 - 0.1^2 = 2 (a^2 - 0.5^2)$	**M1**	Find amplitude *a* from ratio 2 of [½ *m*] v_A^2 to [½ *m*] v_B^2
	$a^2 = 0.5 - 0.01 = 0.49$, $a = 0 .7$ [m]	**A1**	(taking ratio ½ loses A1)
		3	

Question	Answer	Marks	Guidance
2(i)	$2mv_A + mv_B = 2mu$ (AEF)	**M1**	Use momentum (allow *m* omitted)
	$v_B - v_A = ⅔ u$	**M1**	Use Newton's law (M0 if LHS signs inconsistent)
	$v_A = 4 u / 9$, $v_B = 10 u / 9$	**A1, A1**	Combine to find speeds of *A* and *B* after collision
		4	

Question	Answer	Marks	Guidance
2(ii)	$w_B = [-] \frac{1}{2} v_B$ $[= \pm 5u/9]$	**M1**	Relate speed w_B of B after colln. with wall to v_B (ignore sign)
	EITHER: $(d-x)/v_A = d/v_B + x/w_B$ (AEF)	**M1**	*EITHER:* Equate times in terms of dist. x from wall to 3rd colln.
	$(d-x)/4 = d/10 + x/5$, $x = d/3$	**M1A1**	Substitute for speeds to solve for x
	$t = (d-x)/v_A = (2d/3)/(4u/9) = 3d/2u$	**A1**	and hence find reqd. time t
	OR: $x_A = (d/v_B)\, v_A = (9d/10u)/(4u/9) = 2d/5$	**(M1**	*OR:* Find dist. x_A moved by A when B reaches wall
	$t = d/v_B + (d - x_A)/(v_A + w_B)$ $= d/(10u/9) + (3d/5)/(4u/9 + 5u/9)$ $= 9d/10u + 3d/5u$	**M1A1**	Find t by adding times to and from wall (or equivalent method)
	$= 3d/2u$	**A1)**	
		5	

Question	Answer	Marks	Guidance
3(i)	$\frac{1}{2}mv_B^2 = \frac{1}{2}mu^2 - mga(\sin\alpha - \cos\alpha)$ $[v_B^2 = u^2 - (14/13)ag]$	**M1A1**	Find speed v_B at B by conservation of energy (A0 if no m) $[\sin\alpha = 12/13, \cos\alpha = 5/13]$
	$[T_B =]$ $mv_B^2/a - mg\sin\alpha = 0$ $[v_B^2 = (12/13)ag]$	**B1**	Equate tension T_B at B to zero by using $F = ma$ radially
	$u^2 = (3\sin\alpha - 2\cos\alpha)ag$	**M1**	Combine to verify u^2
	$= (36/13 - 10/13)ag = 2ag$ AG	**A1**	
		5	

Question	Answer	Marks	Guidance
3(ii)	*EITHER:* $\frac{1}{2}mv_C^2 = \frac{1}{2}mu^2 + mga\,(1 + \cos\alpha)$ *or* $\frac{1}{2}mv_B^2 + mga\,(1 + \sin\alpha)$ $[v_C^2 = 62ag/13]$	M1	Find v_C^2 at lowest point *C* by conservation of energy
	$T_{max} = mv_C^2/a + mg$	B1	Find tension T_{max} at lowest point from $F = ma$ radially
	$= 62mg/13 + mg$	M1	Combine to find T_{max}
	$= 75mg/13$ *or* $5.77\ mg$ *or* $57.7\ m$	A1	
	OR: $\frac{1}{2}mV^2 = \frac{1}{2}mu^2 + mga\,(\cos\alpha + \cos\theta)$ $[V^2 = 2ag\,(18/13 + \cos\theta)]$	(M1	Find V^2 at general point by conservation of energy (where *OP* is e.g. at θ to downward vertical)
	$T = mV^2/a + mg\cos\theta = (36/13 + 3\cos\theta)\,mg$	B1	Find tension *T* at general point from $F = ma$ radially
	$T_{max} = (36/13 + 3)\,mg$	M1	Combine to find T_{max} at lowest point where $\theta = 0$
	$= 75mg/13$ *or* $5.77\ mg$ *or* $57.7\ m$	A1)	
		4	

Question	Answer	Marks	Guidance
4(i)	A: $R_C \times x - W \cos 45° \times a = 0$ B: $F_A \cos 45° \times 2a - R_A \cos 45° \times 2a - R_C \times (2a - x)$ $+ W \cos 45° \times a = 0$ C: $F_A \cos 45° \times x \quad - R_A \cos 45° \times x + W \cos 45° \times (x - a) = 0$ G: $F_A \cos 45° \times a \quad - R_A \cos 45° \times a + R_C \times (x - a) = 0$ D: $R_A \cos 45° \times a \quad - R_C \cos 45° \times x \cos 45°$ $- R_C \cos 45° \times (x - a) \cos 45° = 0$	B1	Take moments for rod about one chosen point ($F_A \times$ may be replaced by μR_A and cos 45° by e.g. $1/\sqrt{2}$) (A single resolution along the rod will then suffice since no R_C) (G is mid-point of AB) (D is on gound below G)
	Horizontally: $F_A - R_C \cos 45° = 0$	B1	Find two more indep. eqns, e.g. resolution of forces on rod
	Vertically: $R_A + R_C \cos 45° - W = 0$	B1	(a second moment eqn. may be used)
	Along rod: $F_A \cos 45° + R_A \cos 45° - W \cos 45° = 0$	(B1	
	Perp. to rod: $F_A \cos 45° - R_A \cos 45° - R_C + W \cos 45° = 0$	B1)	
	$[R_A = W / (1+\mu),\ F_A = \mu W / (1+\mu),\ R_C = \mu W \sqrt{2} / (1+\mu)]$ $x = a(1+\mu) / 2\mu$ *or* $\frac{1}{2} a (1 + 1/\mu)$	M1A1	Combine to find x (using $F_A = \mu R_A$ and cos 45° = $1/\sqrt{2}$)
		5	
4(ii)	$a(1+\mu) / 2\mu \leqslant 2a$ so $\mu \geqslant \frac{1}{3}$ AG	M1A1	Verify μ using $x \leqslant 2a$
		2	
4(iii)	$a(1+\mu) / 2\mu = 3a/2$ so $\mu = \frac{1}{2}$	M1A1	Find μ when $x = 3a/2$ using result in **(i)**
	$F_A = W/3,\ R_A = 2W/3\ [R_C = (\sqrt{2})\ W/3]$ $N_A = \sqrt{(F_A^2 + R_A^2)} = (\sqrt{5}/3)\ W$ *or* $0.745\ W$	M1A1	Find F_A, R_A and hence magnitude of resultant force N_A at A
		4	

Question	Answer	Marks	Guidance
5(i)	$I_{disc} = ½ \times ½\ 3M\ (2a)^2$ [$= 3\ Ma^2$]	**M1**	Find MI of disc parallel to axis l at its centre using perpendicular axis theorem (M0 if theorem not used)
	$I_{disc}{}' = I_{disc} + 3M\ (5a)^2 = 78\ Ma^2$	**M1A1**	Find MI of disc about axis l
	$I_{rod} = ⅓\ kM\ (3a/2)^2 + kM\ (3a/2)^2 = 3k\ Ma^2$	**B1**	Find or state MI of rod AB about axis l
	$I = (78 + 3k)\ Ma^2 = 3\ Ma^2\ (26 + k)$ AG	**A1**	Find MI of object about axis l (A0 if inadequate explanation)
	SC: $I_{disc}{}' = ½\ \{½\ 3M\ (2a)^2 + 3M\ (5a)^2\} = (81/2)\ Ma^2$	**(M1)**	**SC**: Invalidly applying theorems in wrong order to disc (max 2)
		5	
5(ii)	[–] $I\ \mathrm{d}^2\theta/\mathrm{d}t^2 = 3Mg \times 5a \sin\theta + (kMg \times (3a/2) \sin\theta$ [$= 3\ (5 + ½\ k)\ Mga \sin\theta$]	**M1A1**	Use eqn of circular motion to find $\mathrm{d}^2\theta/\mathrm{d}t^2$ where θ is angle of rod with vertical
	$\mathrm{d}^2\theta/\mathrm{d}t^2 = -\ \{g\ (10 + k)\ /\ 2a(26 + k)\}\ \theta$ (AEF)	**M1A1**	Approximate $\sin\theta$ by θ to show SHM (M0 if $\cos\theta \approx \theta$ used or sign wrong; A0 if LHS unclear)
	$2\pi\ /\ \sqrt{\{\ (10 + k)\ /\ 2(26 + k)\}} = 4\pi$	**M1**	Find k by equating period T to $4\pi\sqrt{(a/g)}$ [so $\omega = ½\sqrt{(g/a)}$]
	$2\ (26 + k) = 4\ (10 + k)$, $k = 6$	**A1**	
		6	

Question	Answer	Marks	Guidance
6	$\bar{x} = 90.3 / 8 = 11.2875$ (to 4 s.f.)	**B1**	Find sample mean
	$s^2 = (1043.67 - 90.3^2/8) / 7$	**M1**	Estimate population variance
	$= 19\,527 / 5600$ *or* 3.487 [*or* 1.867^2] (to 4 s.f.)	**A1**	(allow biased here: 3.051 *or* 1.747^2)
	$90.3 / 8 \pm t\,\sqrt{(s^2/8)}$	**M1**	Find confidence interval
	$t_{7,\,0.975} = 2.365$ (to 4 s.f.)	**A1**	State or use correct tabular value of t
	11.3 ± 1.6 *or* [9.7, 12.8[5]]	**A1**	Evaluate C.I. (either form)
		6	

Question	Answer	Marks	Guidance
7(i)	*EITHER:* $G(y)$ [$= P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$] $= (1/90)\,(y + y^2)$	**M1A1**	Find or state $G(y)$ for $0 \leqslant x \leqslant 3$ from $Y = X^2$ (allow $<$ or $\leqslant$ throughout)
	OR: Use $x = y^{1/2}$ to find $f(x) = (1/90)\,(2x + 4x^3) = (1/90)\,(2y^{1/2} + 4y^{3/2})$ *and* $dx/dy = 1 / 2y^{1/2}$	**(M1A1)**	Find $f(x)$ *and* dx/dy for use in $g(y) = f(x) \times \lvert dx/dy \rvert$
	$g(y)$ [$= G'(y)$] $= (1/90)\,(1 + 2y)$	**A1**	Find $g(y)$ in simplified form
	for $0 \leqslant y \leqslant 9$ [$g(y) = 0$ otherwise]	**A1**	State corresponding range of y at any stage
		4	
7(ii)	$E(Y) = (1/90) \int (y + 2y^2)\,dy$	**M1**	Find mean of Y from $\int y\,g(y)\,dy$
	$= (1/90)\,[\frac{1}{2}y^2 + \frac{2}{3}y^3]_0^9 = 117/20$ *or* 5.85	**A1**	
		2	

Question	Answer	Marks	Guidance
8(i)	$P(X \geqslant 5) = (1 - p)^4 = 0.4096$ $p = 1 - 0.8 = 0.2$ AG	**M1A1**	Verify p using $P(X \geqslant 5) = 0.4096$
		2	
8(ii)	$P(X = 6) = (1 - p)^5 p = 0.8^5 \times 0.2 = 0.0655$	**M1A1**	Find $P(X = 6)$
		2	
8(iii)	$1 - (1 - p)^N > 0.9$	**M1**	Formulate condition for N ($1 - (1 - p)^{N-1}$ is M0)
	$0.1 > 0.8^N$	**A1**	(< or = can earn M1 M1 only, max 2/4)
	$N > \log 0.1 / \log 0.8 = 10.3$	**M1**	Rearrange and take logs (any base) to give bound
	$N_{\min} = 11$ corresponding to Monday of 3rd week	**A1**	Find $N_{\min}$ and corresponding day and week
		4	

Question	Answer	Marks	Guidance
9(i)	H_0: $\rho = 0$, H_1: $\rho \neq 0$	**B1**	State both hypotheses (B0 for r …)
	EITHER: $r_{5,\,5\%} = 0.878$	***B1**	State or use correct tabular two-tail r-value
	Accept H_0 if $\lvert r \rvert <$ tab. value (AEF)	**M1**	State or imply valid method for conclusion
	OR: $t_r = r\sqrt{((n-2) / (1 - r^2))} = -2.08$, $t_{4,0.975} = 2.776$ *or* 2.78	**(*B1**	(Rarely seen)
	Accept H_0 if $\lvert t_r \rvert <$ tab. t-value (AEF)	**M1)**	
	No evidence of [non-zero] correlation (AEF)	**A1**	Correct conclusion (dep *B1)
		4	

Question	Answer	Marks	Guidance
9(ii)	$cd = r^2$ [$= (-0.7214)^2 = 0.52042$]	**M1**	Find cd
	$\bar{x} = 10.8 + d\,(4.2 + c\,\bar{x})$ [$\bar{y} = 2.137$]	**M1**	Find 2nd eqn. for c, d using <u>means</u> in eqns. of regression lines Combine to find c (or d)
	$c = 4.2\,r^2 / \{(1 - r^2)\bar{x} - 10.8\} = -0.294$ *or* $d = \{(1 - r^2)\bar{x} - 10.8\} / 4.2 = -1.77$	**M1A1**	
	$d = 0.7214^2 / c = -1.77$ *or* $c = 0.7214^2 / d = -0.294$	**A1**	and hence d (or c)
		5	
9(iii)	$x = 10.8 - 1.77 \times 3.5 = 4.60[5]$ [y on x gives 2.38]	**B1**	Find y from eqn. of regression line of x on y
	e.g. not reliable since no evidence of correlation *or* reasonably reliable since 0.7214 close to 1 *or* not reliable since 0.7214 not close to 1 *or* reliability unclear as degree of extrapolation unknown	**B1**	Valid comment on reliability (AEF)
		2	

Question	Answer	Marks	Guidance
10(i)	$\bar{x} = \Sigma\, x\, \mathrm{f}(x) = 118/40 = 2.95$ AG	**B1**	Verify mean of sample data (B0 for $\bar{x} = 118/40 = 2.95$)
	$\sigma_n^2 = (454 - 118^2/40) / 40 = 2.65$ *or* $\sigma_{n-1}^2 = (454 - 118^2/40) / 39 = 2.72$	**B1**	Find variance of sample data (accept either σ_n^2 or σ_{n-1}^2)
	$2.95 \approx 2.65$ *or* 2.72	**B1**	Valid comment on correct values
		3	

Question	Answer	Marks	Guidance
10(ii)	$E_4 = 40\ \lambda^4\ e^{-\lambda}/4!$ with $\lambda = 2.95$ [$= 40 \times 0.1652$] (allow $(\lambda/4) \times E_3 = 0.7375 \times 8.96$)	**M1A1**	State expression for reqd. expected value E_4
		2	
10(iii)	H_0: [Poisson] distribution fits data (AEF)	**B1**	State (at least) null hypothesis in full
	O_i: <u>8</u> 8 10 5 <u>9</u> E_i: <u>8.27</u> 9.11 8.96 6.61 <u>7.05</u>	**M1**	Combine values consistent with all exp. values ⩾ 5
	$X^2 = 0.0088 + 0.1352 + 0.1207 + 0.3921 + 0.5394$	**M1**	Find value of X^2 from $\Sigma (E_i - O_i)^2 / E_i$ [or $\Sigma O_i^2/E_i - n$]
	$= 1.20$ (to 3 s.f.)	**A1**	
	No. n of cells: 8 7 6 <u>5</u> $\chi_{n-2,\ 0.95}{}^2$: 12.59 11.07 9.488 <u>7.815</u>	**B1**	State or use consistent tabular value $\chi_{n-2,\ 0.95}{}^2$ (to 3 s.f.) [**FT** on number, n, of cells used to find X^2]
	Accept H_0 if X^2 < tabular value (AEF)	**M1**	State or imply valid method for conclusion
	1.20 [± 0.1] < 7.81[5] so [Poisson] distn. fits [data] *or* distn. is a suitable model (AEF)	**A1**	Conclusion (requires both values correct)
		7	

Question	Answer	Marks	Guidance
11A(i)	*EITHER:* $40\,e_0 / 0.8 = 2g$, $e_0 = 0.4$ [*or* $OP_0 = 1.2$] [m]	**M1A1**	Find extension e_0 [or OP_0] at equilibrium position P_0
	$2\,\mathrm{d}^2x/\mathrm{d}t^2 = -40\,(e_0 + x) / 0.8 + 2g$ *or* $= +40\,(e_0 - x) / 0.8 - 2g$	**M1A1**	Use Newton's law at general point (e.g. x below or above P_0) (ignore LHS sign here only)
	OR: $2\,\mathrm{d}^2y/\mathrm{d}t^2 = -40\,(y - 0.8) / 0.8 + 2g$ $= -50\,y + 60$	**(M1A1**	Use Newton's law at general point in terms of $y = OP$ (ignore LHS sign here only)
	Take $x = y - 1.2$ to give	**M1A1)**	Change variable to give standard form of SHM eqn
	$\mathrm{d}^2x/\mathrm{d}t^2 = -25x$	**A1**	Hence SHM (A0 if wrong sign or LHS unclear) (B1 only for stating SHM eqn. without proof)
	$T = 2\pi / \sqrt{(25)} = 2\pi/5$ AG	**A1**	Verify period T using $T = 2\pi/\omega$ with $\omega = \sqrt{(25)}$
	$OP_0 = 1.2$ [m]	**B1**	State OP_0 explicitly (may imply first M1 A1)
		7	
11A(ii)	$0.4^2 = 25\,(a^2 - 0.06^2)$	**M1A1**	Find amplitude a from $v^2 = \omega^2\,(a^2 - x^2)$
	$a = \sqrt{(0.0064 + 0.0036)} = 0.1$ [m]	**A1**	
		3	
11A(iii)	$40\,e_1 / 0.8 = (2 + M)\,g$, [$e_1 = 0.4 + 0.2\,M$] *or* $40(e_1 - e_0) / 0.8 = Mg$ [$e_1 - e_0 = 0.2\,M$]	**M1A1**	Find extension e_1 [or OP_1] at first equilibrium position P_1 *or* equate additional extension to M by Hooke's Law
	$e_1 - e_0 = a$, $M = a/0.2 = 0.5$	**M1A1**	Find M by relating e_0, e_1 and a
		4	

Question	Answer	Marks	Guidance
11B(i)	H$_0$: $\mu_x = \mu_y$, H$_1$: $\mu_x > \mu_y$ (or in terms of μ_A , μ_B)	**B1**	State hypotheses (B0 for $\bar{x}$ …)
	$s_x^2 = (14.1775 - 10.56^2/8) / 7 = 0.03404$ *and* $s_y^2 = (15.894 - 12.39^2/10) / 9 = 0.06031$ (to 3 s.f.)	**M1A1**	Estimate both population variances (allow biased here: 0.02979 *and* 0.05428)
	$s^2 = (7\,s_x^2 + 9\,s_y^2) / 16$ *or* $(14.1775 - 10.56^2/8 + 15.894 - 12.39^2/10) / 16$ $= 0.78\,109 / 16$ *or* 0.04882 *or* 0.22095^2 (to 3 s.f.)	**M1A1**	Find pooled estimate of common variance (M1 A1 for s_x^2 and s_y^2 may be implied here)
	$t_{16,\,0.9} = 1.337$ (to 3 s.f.)	***B1**	State or use correct tabular t value
	$t = (1.32 - 1.239) / s\sqrt{(1/10 + 1/8)} = 0.773$	**M1A1**	Calculate value of t (or $-t$) (or can compare $\bar{y} - \bar{x} = 0.081$ with 0.140)
	$t <$ tabular value so claim not justified *or* *A*'s not heavier than *B*'s (AEF)	**B1**	Correct conclusion (**FT** on t, dep *B1)
	SC: $z = (1.32 - 1.239) / \sqrt{(s_x^2/8 + s_y^2/10)}$ $= 0.081 / \sqrt{(0.078)} = 0.799$	**(B1**	**SC:** Implicitly taking s_x^2, s_y^2 as unequal popln. variances (may also earn first B1 M1 A1)
	$z < 1.282$ so claim is not justified (AEF)	**B1)**	Comparison with $z_{0.9}$ and conclusion (**FT** on z; max 5/9)
		9	
11B(ii)	$\bar{x} = 1.28$ *and* $s^2 = 0.294 / 7$ [$= 0.042$ (*or* 0.205^2)]	**M1**	Find sample mean & estimate popn. var (allow M1 if 0.294 / 8)
	$t = (1.28 - p) / \sqrt{(0.042/8)}$ (AEF)	**M1A1**	Find value of t
	$t > 1.415$ (< is A0), $p < 1.28 - 0.1025$, $p_{max} = 1.18$	**M1A1**	Find p_{max} by comparison with tabular value, here $t_{7,\,0.9}$
		5	