1 Find the eigenvalues of the matrix

$$\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}.$$

Find also corresponding eigenvectors. [5]

2 Given that

$$C_n = \int_0^1 (1-x)^n \cos x \, dx \quad \text{and} \quad S_n = \int_0^1 (1-x)^n \sin x \, dx,$$

show that, for $n \geqslant 1$,

$$C_n = nS_{n-1} \quad \text{and} \quad S_n = 1 - nC_{n-1}.$$ [3]

Hence find the value of S_3, correct to 6 decimal places. [3]

3 Given that

$$S_N = \sum_{n=1}^{N} (2n-1)^3,$$

show that

$$S_N = N^2(2N^2 - 1).$$ [4]

Hence find

$$\sum_{n=N+1}^{2N} (2n-1)^3$$

in terms of N, simplifying your answer. [3]

4 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 15x + 16.$$ [6]

Show that, whatever the initial conditions, $y \approx 3x + 2$ when x is large and positive. [1]

5 The roots of the equation $x^3 - 3x^2 + 1 = 0$ are denoted by α, β, γ. Show that the equation whose roots are

$$\frac{\alpha}{\alpha - 2}, \frac{\beta}{\beta - 2}, \frac{\gamma}{\gamma - 2}$$

is $3y^3 - 9y^2 - 3y + 1 = 0$. [3]

Find the value of

(i) $(\alpha - 2)(\beta - 2)(\gamma - 2)$, [3]

(ii) $\alpha(\beta - 2)(\gamma - 2) + \beta(\gamma - 2)(\alpha - 2) + \gamma(\alpha - 2)(\beta - 2)$. [2]

6 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 4$ and

$$u_{n+1} = \frac{5u_n + 4}{u_n + 2}.$$

By considering $4 - u_{n+1}$, or otherwise, prove by induction that $u_n < 4$ for all $n \geqslant 1$. [5]

Prove also that $u_{n+1} > u_n$ for all $n \geqslant 1$. [3]

7 Given that

$$x = 1 + \frac{1}{t} \quad \text{and} \quad y = t^3 e^{-t},$$

where $t \neq 0$, find $\frac{dy}{dx}$ in terms of t. [3]

Find $\frac{d^2y}{dx^2}$ in terms of t, and hence find the values of t for which $\frac{d^2y}{dx^2} = 0$. [5]

8 The arc of the curve with equation

$$y = \tfrac{3}{8}x^{\frac{4}{3}} - \tfrac{3}{4}x^{\frac{2}{3}},$$

from the point where $x = 1$ to the point where $x = 8$, is denoted by C.

(i) Find the length of C. [5]

(ii) Find the area of the surface generated when C is rotated through one revolution about the y-axis. [3]

9 Given that $w_n = 3^{-n} \cos 2n\theta$ for $n = 1, 2, 3, \ldots$, use de Moivre's theorem to show that

$$1 + w_1 + w_2 + w_3 + \ldots + w_{N-1} = \frac{9 - 3\cos 2\theta + 3^{-N+1}\cos 2(N-1)\theta - 3^{-N+2}\cos 2N\theta}{10 - 6\cos 2\theta}.$$ [7]

Hence show that the infinite series

$$1 + w_1 + w_2 + w_3 + \ldots$$

is convergent for all values of θ, and find the sum to infinity. [2]

10 The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are given by

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The subspace of $\mathbb{R}^4$ spanned by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is denoted by V_1, and the subspace of $\mathbb{R}^4$ spanned by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ is denoted by V_2. Show that V_1 and V_2 each have dimension 3. [3]

The set of vectors which belong to both V_1 and V_2 is denoted by V_3. Find a basis for V_3. [2]

The set of vectors which consists of the zero vector and all vectors which belong to only one of V_1 and V_2 is denoted by W.

(i) Write down two linearly independent vectors which belong to W. [2]

(ii) Show that W is not a linear space. [3]

11 The line l_1 passes through the points with position vectors $2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. The line l_2 has equation $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} - 11\mathbf{k} + t(5\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The line l_3 is perpendicular to l_1 and l_2, and passes through the point with position vector $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(i) Find the equation of the plane which contains l_1 and l_3, giving your answer in the form $ax + by + cz = d$. [5]

(ii) Show that l_2 and l_3 intersect. [4]

(iii) Find the shortest distance between l_1 and l_2. [2]

Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{a(x-a)^2}{x^2 - 4a^2},$$

where a is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C has one maximum point and one minimum point and find their coordinates. [6]

(iii) Sketch C, and give the coordinates of any points where C meets the axes. [4]

OR

The curve C has polar equation

$$r = a(1 + \cos\theta), \qquad -\pi < \theta \leqslant \pi,$$

where a is a positive constant.

(i) Sketch C. [2]

(ii) Show that the area of the region enclosed by C is $\frac{3}{2}\pi a^2$. [5]

(iii) The point on C with polar coordinates (r, θ) has cartesian coordinates (x, y). Find the minimum value of y. [6]