FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

- Candidates should take great care when sketching graphs to show the sections of the graph that match the given domain. Sketches should be drawn and labelled carefully
- Candidates should use the specified method if instructed.
- They should ensure they show all the steps needed to support their solution, particularly when proving a given result

General comments

The majority of candidates showed very good knowledge across the whole syllabus, and took care over their calculations avoiding loss of marks through careless errors.

Question 1

Most candidates obtained full marks on this question, recalling the formula for arc length and recognising the integrand as the square root of a perfect square. A very small number of candidates used the polar version of the formula.

Answer: $2 + e^3$

Question 2

Although most candidates showed a good knowledge of the structure of an induction proof, some did not communicate all the steps clearly. Common errors included using the wrong base case, failing to explain why f(k) was divisible by 9 and not making a final statement. Some candidates correctly defined a hypothesis, H_k and were able to refer to this during their proof. This avoided statements about f(k) 'being true' being used in place of f(k) being divisible by 9. A small number of candidates produced a deductive proof. Most candidates manipulated the expressions well.

Question 3

(i) Although most candidates knew the shape of this graph, some failed to take note of the domain of θ so produced extra sections. Candidates must remember to mark key points, such as the intersection with the initial line, and should take care to show any symmetry. They should also ensure that their sketches show clearly the behaviour of curves at their extremities.

(ii) The majority of candidates were able to recall and apply the formula for finding the area of a polar curve, and use the double angle formula correctly.

(iii) Many candidates were able to use a double angle formula here too, and then substitute for x and y.

Answers: **(ii)** $\frac{\pi}{8}$ **(iii)** $\left(x^2 + y^2\right)^{\frac{3}{2}} = x^2 - y^2$

Question 4

(i) (ii) This question was very well done by most candidates. They recalled the relationships between the roots and coefficients of an equation and were able to use them to find the roots and k.

Answers: **(i)** 3, 6, 12 **(ii)** 126

Question 5

(i) Candidates who started by writing out the first few terms of the series were usually able to see the necessary summations. This is an effective approach to such questions, where the $(-1)^n$ can otherwise cause confusion.

(ii) Although most candidates were able to find the first limit, many failed to appreciate the effect of the $(-1)^n$ term here, and tried to apply the summation formula (valid for 2n) from **(i)** again. Those who realised that they only needed to add the next term to S_{2n} were usually able to find the second limit.

Answers: **(ii)** –2 and 2

Question 6

(i) Apart from a small number of errors in division, the first part of the question was well done.

(ii) (iii) A number of candidates failed to explain the significance of $b > 0$ when justifying that there is no x intercept. In finding the number of stationery points, most candidates differentiated correctly resulting in a quadratic expression which they set equal to zero. Unfortunately, a significant number forgot to use the discriminant to show that there were two roots to this equation, and just assumed that the quadratic equation did have two roots.

(iv) Stronger candidates produced well drawn and labelled sketches, but others did not show correct forms at infinity, with curves bending away from asymptotes. Some sketches were missing branches, but most did have the general shape correct.

Answers: **(i)** $y = x - b$ and $x = -b$ **(iii)** $b^2 + b > 0$ so 2 turning points.

Question 7

This question was very well done by the majority of candidates, who recognised the correct form of both the complementary function and the particular integral. There were some slips in finding the particular solution, but most candidates worked through the steps very competently.

Answer: $y = (-49 - 56x)e^{-\frac{1}{7}x} + 49(x+1)$

Question 8:

(i) Most candidates attempted this question by reducing the matrix to row echelon form before forming and solving a system of equations. Unfortunately many made an error in the process resulting in an extra vector in the basis of K_1. A few lost generality by assigning a value to α.

(ii) Setting $\alpha = 0$ reduced the number of errors, so most candidates correctly found a basis for K_2.

(iii) Errors in the first part contributed to difficulties in this part. Stronger candidates realised that the K_1 was a subspace of K_2 because its basis was a subset of the basis of K_2.
Candidates need to be careful to distinguish between the basis of a vector space and the space itself.

Answers: **(i)** $K_1 = \left\{\begin{pmatrix}0\\1\\0\\2\end{pmatrix}\right\}$ **(ii)** $K_2 = \left\{\begin{pmatrix}-6\\3\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix}\right\}$ or equivalent **(iii)** K_1 is a subspace of K_2

Question 9

(i) The first part of this question was well done, with the majority of candidates using the given substitution. Some forgot to write their final answer in terms of *x*. A few realised that they could integrate directly.

(ii) Candidates who realised the connection between parts **(i)** and **(ii)** split the integral correctly to make use of the result of **(i)** and proceeded to the correct final answer usually, using a trigonometric identity to form the reduction formula.

(iii) Although most candidates were able to use the reduction formula successfully, several did not use I_4 to calculate the mean value.

Answers: **(i)** $\frac{1}{3}\tan^3 x + c$ **(iii)** Mean value $= \frac{4}{\pi}I_4 = \frac{32}{15\pi}$

Question 10

(i) (a) This part was very well done by most candidates

(b) Apart from a number of numerical slips in the calculation of the vector product, solutions were generally good, with candidates knowing how to find the normal to the plane and then using a point on the plane to find the value of the constant. Some used the point of intersection, but most chose a point on one of the lines.

(ii) Here again, some calculation errors in vector or scalar products (or both!), detracted from generally sound solutions. Most candidates used the standard formula to good effect. A very small number assumed that the cross product produced the shortest distance directly.

Answers: **(i)(b)** $-13x + 5y - z = 43$ **(ii)** $a = 3$

Question 11

EITHER:

(i) Candidates tackled this in a variety of ways, usually using the exponential form of a complex number. Other solutions used $z = \cos(\theta) + i\sin(\theta)$, finishing off by using trigonometric identities to reach the correct expression.

(ii) Most candidates used the result from the first part to re-write the given equation in terms of tangents, though others approached the problem algebraically. These were able to prove the result generally, whilst those using tangents needed to justify their answer for each of the cube roots of unity.

(iii) Very few candidates attempted this, though some good solutions using half angles, as well as a more algebraic approach were seen.

Answers: **(iii)** $z = 1,\ e^{\pm i\frac{2\pi}{3}},\ -1,\ \frac{1}{3} \pm i\frac{2}{3}\sqrt{2}$

OR:

This was very much the more popular choice.

(i) (ii) Few candidates realised that a multiple of **e** is also an eigenvector, rather more remembered that λ^n is the eigenvalue of $\mathbf{A}^n$ and that **e** was an eigenvector.

(iii) The majority of candidates made good progress with this part of the question, though they could have saved themselves time by recognising the eigenvalues from the echelon form of the matrix. Some errors were made with finding eigenvectors, and some candidates forgot to raise the values of λ to the power of *n* in the matrix **D.**

(iv) Strong candidates recognised that this question required use of the method of differences enabling them to make progress, though not all recognised the constraint on the elements of **D.**

Answers: **(i)** **e.g.** $2\mathbf{e}$ **(ii)** Eigenvector: $\mathbf{e}$, Eigenvalue: λ^n

(iii) $\mathbf{P} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 7^n & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **(iv)** $-\frac{1}{7} < k < \frac{1}{7}$.

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

Key messages

- Candidates should take great care when sketching graphs to show the sections of the graph that match the given domain. Sketches should be drawn and labelled carefully
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(iv) Stronger candidates produced well drawn and labelled sketches, but others did not show correct forms at infinity, with curves bending away from asymptotes. Some sketches were missing branches, but most did have the general shape correct.

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(i) Most candidates attempted this question by reducing the matrix to row echelon form before forming and solving a system of equations. Unfortunately many made an error in the process resulting in an extra vector in the basis of K_1. A few lost generality by assigning a value to α.

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Answers: **(i)** $\frac{1}{3}\tan^3 x + c$ **(iii)** Mean value $= \frac{4}{\pi}I_4 = \frac{32}{15\pi}$

Question 10

(i) (a) This part was very well done by most candidates

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Question 11

EITHER:

(i) Candidates tackled this in a variety of ways, usually using the exponential form of a complex number. Other solutions used z = cos(θ) + isin(θ), finishing off by using trigonometric identities to reach the correct expression.

(ii) Most candidates used the result from the first part to re-write the given equation in terms of tangents, though others approached the problem algebraically. These were able to prove the result generally, whilst those using tangents needed to justify their answer for each of the cube roots of unity.

(iii) Very few candidates attempted this, though some good solutions using half angles, as well as a more algebraic approach were seen.

Answers: **(iii)** $z = 1,\ e^{\pm i\frac{2\pi}{3}},\ -1,\ \frac{1}{3} \pm i\frac{2}{3}\sqrt{2}$

OR:

This was very much the more popular choice.

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(iii) The majority of candidates made good progress with this part of the question, though they could have saved themselves time by recognising the eigenvalues from the echelon form of the matrix. Some errors were made with finding eigenvectors, and some candidates forgot to raise the values of λ to the power of *n* in the matrix **D.**

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Answers: **(i)** **e.g.** $2\mathbf{e}$ **(ii)** Eigenvector: $\mathbf{e}$, Eigenvalue: λ^n

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FURTHER MATHEMATICS

Paper 9231/13
Paper 13

Key messages

- Candidates should ensure they show clearly all the steps needed to justify their solutions, particularly when working towards a given answer.
- Candidates should take great care when sketching graphs to show the sections of the graph that match the given domain. Sketches should be drawn neatly and labelled carefully.
- Candidates should use a particular method when it is specified in the question.
- They should be able to recall and apply skills from the 9709 syllabus where appropriate.

General comments

Most candidates attempted all the questions, and were able to show their knowledge and skills to good advantage. Those who took care over accuracy as well as showing knowledge of techniques scored very well, with many producing elegant solutions throughout the paper. Some graph sketching was excellent, but some scripts showed less attention to detail, and less care over drawing. Algebra and calculus skills of a high standard were shown in many scripts.

Comments on specific questions

Question 1

(i) The first part of this question did not pose any difficulties.

(ii) Candidates who used the chain rule were usually successful. Those who chose to multiply out the expression were at risk of errors, particularly if they multiplied the brackets out rather than using a binomial expansion. A small number of candidates took the cube root of both sides of the equation before differentiating, and were usually successful too.

Answers: **(i)** $\frac{dy}{dx} = 1$ **(ii)** $\frac{d^2y}{dx^2} = -\frac{4}{3}$

Question 2

Most candidates were able to verify that the given expressions in part **(i)** were equivalent, using variety of algebraic methods, including but not exclusively partial fractions. They went on to use the difference method in part **(ii)** to find the Nth sum without difficulty. Care must be taken over notation – capital N was required here.

(iii) Most candidates then used this to identify *S* correctly, and were able to solve the given inequality. Most candidates completed the question successfully by identifying the least integer value for N that satisfied the inequality, though a few left their answer as a decimal, thus losing the final mark. In general, candidates were able to demonstrate their skills in handling inequalities and expressions involving logarithms.

Answers: **(ii)** $S_N = \frac{1}{e} - \frac{1}{(N+1)e^{N+1}}$ **(iii)** 6

Question 3

(i) Most candidates used de Moivre's Theorem as required to verify the expression for $\cos(4\theta)$, with a very small number wrongly attempting the verification using trigonometric identities alone.

(ii) Whilst many candidates then correctly divided the expression from **(i)** by $\cos^4(\theta)$ to produce a quartic in $\tan(\theta)$, others divided by $\sin^4(\theta)$ but failed to realise that this leads to an expression in $\cot(\theta)$. Some failed to make a link with the first part as required. A variety of correct methods, some involving $\cot(4\theta)$, but most arising from $\cos(4\theta) = 0$, were seen. Whilst most candidates made sure their answers were in the correct form, and were distinct, a small number gave repeated roots or failed to match the required form, losing marks.

Answers: **(ii)** $\tan q\pi$ where $q = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ or equivalent.

Question 4

(i), (ii) Most candidates found the intersection points without difficulty and were able to identify the vertical asymptote. Some errors in polynomial division were seen as candidates found the equation of the oblique asymptote.

(iii) Whilst most candidates sketched the asymptotes and placed the two branches of the curve correctly, marks were lost from poor drawing particularly with curves that did not tend to asymptotes as they should have done. Some candidates only drew one branch of the curve, or only showed a very small portion of one branch. All key points and lines should be clearly labelled and care should be taken with drawing the curves as they approach the asymptotes.

Answers: **(i)** (–6,0), (–1,0), (0,–3) **(ii)** $x = 2$ and $y = x + 9$

Question 5

(i) Most candidates constructed the proof well, but a number failed to show every step, or reversed the multiplication of A and e, or did not state the resulting eigenvalue as required.

(ii) This part of the question was also well done, though some candidates accepted zero eigenvectors without checking for errors in their working. Some candidates worked from A, others found $A^3 + I$ first, though the latter method sometimes led to errors in finding A^3.

Answers: **(i)** λ^3 **(ii)** $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 9 & 0 \\ 0 & 28 \end{pmatrix}$

Question 6

(i) Most candidates substituted for x into the original equation, others verified the result by substituting for y into the second cubic.

(ii) Apart from a small number of sign errors, most candidates used the relationship $y^3 = 2y + 7$ successfully to find S_3 though a few missed the 3x7 required when summing. Some used a formula for the sum of the cubes of roots, though not all remembered the formula correctly.

(iii) Several methods were employed in the final part, with some candidates opting to find a third cubic with reciprocal roots. Other candidates used $7S_{-2} = S_1 - 2S_{-1}$ or found an expression for S_{-2} in terms of products and squares of products of roots. Some candidates trying to recall complicated sigma formulae did make errors again here.

Answers: **(ii)** 21 **(iii)** $\frac{4}{49}$

Question 7

(i) The first part of this question was very well done, with few errors.

(ii) A variety of methods were used successfully here, with candidates using known formulae to good effect. Others found the coordinates of the foot of the perpendicular from P to the plane and hence its length, working well on a 'first principles' basis.

(iii) This part of the question posed more of a challenge, though many candidates knew how to find the required distance by applying a formula. Others again went from first principles and found the foot of the perpendicular, using the scalar product to find the parameter.
In both parts **(ii)** and **(iii)** where the cross product formula was required, a number of errors were made. Some candidates did not make it clear which vectors they were finding, and would have benefitted from the use of sketches to clarify their methods.

Answers: **(i)** a = 2 **(ii)** $\sqrt{10}$ **(iii)** $\sqrt{10}$

Question 8

(i) Although the majority of candidates found the angle required correctly, it was almost as common to see polar coordinates reversed as written (r , θ) the correct way around. A small number gave angles outside the specified domain.

(ii) Strong candidates produced well drawn and well labelled semi-circles, correctly positioned. Others offered complete circles, shapes that were poorly drawn and no labelling. The graphs were variations on very standard polar curves, so candidates were expected to draw semicircles in the correct position.

(iii) Most candidates were able to integrate correctly, remembering the formula for finding the area, and dealing with the $\cos^2\theta$ in the integrand correctly. Some candidates had difficulty identifying the correct area and used wrong limits in their integration. It is important to label diagrams in order to refer to particular sections of the graph. A number of candidates were able to solve this problem by applying their knowledge of geometry.

Answers: **(i)** $\left(a, \frac{\pi}{3}\right)$ and $\left(a, \frac{2\pi}{3}\right)$ **(iii)** $-\frac{\pi a^2}{6} + \frac{a^2\sqrt{3}}{2}$

Question 9

(i) Although most candidates clearly knew the steps required for an induction proof, sometimes details were omitted. Strong candidates make it clear what they were doing at each stage, and confirmed that they had checked the base case (k = 1 not 2), stated the assumption that the result is true for some k and proved the inductive step in detail. They then summarised by stating that H(k) implied H(k + 1) and that the base case was correct.

(ii) Many candidates realised that this was a geometric series and were able to identify the common ratio but several then forgot that $|r| < 1$, and consequently lost the negative inequality.

(iii) This part of the question was generally well done, with candidates handling the properties of logarithms and the resulting expression accurately. It was a useful strategy to write out the first terms of the summation to get a better idea of its structure – this also helped some candidates with part **(ii)**.

Answers: **(ii)** $-\frac{4}{5} < x < \frac{4}{5}$ **(iii)** $a = \frac{\sqrt{5}}{2}, \quad b = 2\sqrt{5}$ *or equivalent*

Question 10

(i) Most candidates took the most straightforward approach and differentiated y using the chain rule. Those who rearranged to make *x* the subject of the equation were not always able to differentiate correctly, and some took t as a constant. The second method made proving the given transformation of the differential equation more difficult too.

(ii) The majority of candidates used the correct form of the complementary function, as well as the particular integral. Those who stayed with y as a function of t had an easier path to finding the constants than those who reverted to *x* as a function of t early on and errors slipped in.

Answer: **(ii)** $x = \dfrac{\cos 3t - \pi \sin 3t + 9t^2 + 1}{27t}$

Question 11 – **EITHER**

(i) The first part of this question was well done, with few errors. It was pleasing to see a number of students using complex numbers and producing elegant solutions.

(ii) Most candidates were able to find the reduction formula, integrating by parts twice and then realising how a trigonometric identity allowed them to simplify to reach the result. Some candidates omitted to write down the definite integral at each stage – this should be done, even when it is zero.

(iii) Candidates knew the formula for the centroid, and were able to apply the reduction formula successfully.

Answer: **(iii)** $\bar{y} = \dfrac{1}{8}\left(\dfrac{e^{\pi} - e^{-\pi}}{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}\right) = 0.575$

Question 11 – **OR**

This was a slightly less popular choice

(i) The majority of candidates who tackled this option were able to use row operations accurately, although a number did not complete the process. A small number showed that four columns were dependent and that three columns were independent.

(ii) Most candidates were able to find the correct relationship between the column vectors.

(iii) Most candidates gave the first three columns though some found different base vectors.

(iv) It was pleasing to see how many candidates could solve this equation by separating the problem into two parts – finding the particular solution and then identifying the basis for the null space was the most common method. A number of candidates combined the two parts and set up and solved equations or used the augmented matrix. Working with the equations did give rise to errors.

(v) Relatively few candidates gave reasoned responses to this part.

Answers: **(ii)** $\mathrm{v}_4 = \mathrm{v}_3 - \mathrm{v}_2 - 2\mathrm{v}_1$ **(iii)** $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$

(iv) E.g. not closed under addition $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a suitable diagram. Annotating a diagram printed in the question paper may be sufficient if the result is clear, but it may well be preferable to draw a fresh diagram within the answer.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Question 5** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 4** and the directions of motion of particles. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 7, 10 and 11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

Comments on specific questions

Question 1

Although various approaches to finding the mass m of the bullet are possible, a straightforward one is to equate the change in momentum, $m(250 - 70)$, to the impulse causing it, $450 \times 0{\cdot}04$. Another is to find the deceleration of the bullet and apply Newton's second law of motion. Many candidates made a good attempt, but care must be taken over the signs, to avoid the implicit assumption that the bullet reverses direction, for example.

Answer: 0·1.

Question 2

Since P is moving in simple harmonic motion, equating the ratio of its speeds at A and B to 3 : 4 using the standard result $v^2 = \omega^2(a^2 - x^2)$ leads to an equation for a^2, and hence the required value of the amplitude a. The second part similarly requires the use of standard SHM results, namely maximum speed = ωa to give $\omega = \frac{\pi}{6}$ and then $T = \frac{2\pi}{w}$ to give the period T. The required time from A to B may be found in different ways, but in all cases it is important to consider whether the expression $a \sin \omega t$ or $a \cos \omega t$ is the appropriate one for each of the distances being found.

Answers: **(i)** 2 m; **(ii)** 12 s; **(iii)** 3·00 s.

Question 3

Many candidates were able to formulate two equations for the speed v_A of the sphere A after the collision by means of conservation of momentum and Newton's restitution equation, and then eliminate v_A to verify the given expression for e. It was clear that some candidates needed to go back and correct a sign or signs in their initial working in order to achieve the given result. Equating the final kinetic energy $\frac{1}{2}mv_A^2$ to the appropriate proportion of the initial kinetic energy $\frac{1}{2}m(1 + k^2)u^2$ leads to a quadratic equation in k and hence the required value of e. The proportion to be used in this approach is 0·4, but 0·6 was quite often seen. The quadratic equation has of course two roots, and candidates should take care to select the appropriate one, which here means rejecting $k = \frac{1}{3}$ and hence the unacceptable value $-\frac{1}{2}$ for e.

Answer: **(ii)** 3, $\frac{1}{2}$.

Question 4

As in all similar questions, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram so that the symbols used to represent them are clear. Following the instruction in the question, most candidates succeeded in verifying the given magnitude of the normal reaction at B by taking moments about A. A similar process gives the new magnitude after the movement of the particle up the rod, and using the given information about the ratio of the two magnitudes verifies the given value of x. Finally the magnitudes of the frictional and normal reaction forces at A are found by horizontal and vertical resolution of the forces on the rod when the particle is in its second position, and equating their ratio to the given coefficient of friction gives the required value of $\tan \theta$. Taking moments about B, say, is an alternative to a horizontal resolution, but with the disadvantage of more working.

Answers: **(iii)** 1·25.

Question 5

Verifying the given moment of inertia I requires the use of standard formulae and both the perpendicular and parallel axis theorems. The obvious, and popular, approach is to formulate and then sum the individual moments of inertia of the rings about the axis *l,* in which case it is of course necessary to first apply the perpendicular axis theorem, and then the parallel axis theorem, since the reverse order gives an incorrect result. An alternative is to find the moments of inertia of the three rings both about the line of symmetry of the object (intersecting the axis *l* at *O*, say) and about an axis through *O* which is perpendicular to the plane of the rings. The perpendicular axis theorem may then be applied at *O*. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms for I with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. Candidates who initially fail to achieve the given result, whether through not making use of the perpendicular axis theorem, misusing the parallel axis theorem to shift the axis of rotation from a point which is not the centre of mass, or finding a distance incorrectly, should look for their error instead of adding or subtracting a spurious term whose only justification is that it produces the given result, since this is unlikely to satisfy the Examiners. In the second part, the couple acting on the system should be found in terms of $\sin\theta$, where θ is the small angular displacement, and equated to $I\dfrac{d^2\theta}{dt^2}$. Approximating $\sin\theta$ by θ yields the familiar form $\dfrac{d^2\theta}{dt^2} = -\omega^2\theta$ of the standard SHM equation, in which the minus sign is essential, and from which the required period $\dfrac{2\pi}{w}$ is obtained. While this final result may be stated in a number of acceptable forms, there should be some attempt at simplification. Candidates should be very much aware that the constant parameter ω in the SHM equation is not equal to $\dfrac{d\theta}{dt}$ even though the same symbol is sometimes used to denote the latter, and thus finding $\dfrac{d\theta}{dt}$ from conservation of energy at some arbitrary angular displacement and equating it to the SHM parameter ω is wholly invalid.

Answer: **(ii)** $2\pi\sqrt{\dfrac{6a}{g}}$ or $4{\cdot}87\sqrt{a}$.

Question 6

Most candidates produced a correct answer to the first part, by finding $1 - F(2)$. The meaning of interquartile range was not always well understood, and even some candidates who found the values of the upper and lower quartiles correctly, namely 3·466 and 0·719, did not appreciate that the interquartile range means their difference.

Answers: **(i)** 0·449; **(ii)** 2·75.

Question 7

As in all such tests, the hypotheses required in the first part should be stated in terms of the population mean and not the sample mean. The unbiased estimate 27·96 of the population variance may be used to calculate a t-value of 1·69. Since it is a two-tail test, comparison with the tabulated value of 1·943 leads to acceptance of the null hypothesis, so the mean mass of athletes is equal to 94 kg.

Question 8

Most candidates produced good answers to this question. After stating the precise hypotheses to be tested, a table of the expected values for type of car against colour is produced in the usual way and preferably to an accuracy of one or more decimal places. The calculated X^2-value 5·57 should be compared with the tabular value 7·779, leading to acceptance of the null hypothesis. Thus the required conclusion is that the colour of car chosen by customers is independent of the type of car. While some candidates chose to mention the absence of association (or relation) in preference to independence, the latter is preferable in the context of this question.

Question 9

The variance of X was widely known to equal $\frac{q}{p^2}$, where $q = 1 - p$, and equating this to the given value verifies the given quadratic equation. When solving this equation to find the value of p, the inadmissible root –3 must of course be rejected. Most candidates also knew that the probabilities in the next two parts of the question are respectively q^2p and q^3. The starting point in the final part is to formulate the inequality $1 - q^N > 0{\cdot}999$, solution of which gives $N > 4{\cdot}98$ and hence the least integral value of N. It is important to use the correct power of q in the initial inequality, and to reverse the inequality when multiplying through by a negative factor.

Answers: **(i)** $\frac{3}{4}$; **(ii)** $\frac{3}{64}$; **(iii)** $\frac{1}{64}$; **(iv)** 5.

Question 10

As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Many candidates found an unbiased estimate 0·6378 of the combined variance s, leading to a calculated z-value of 1·75. Since it is a one-tailed test, comparison with the tabulated value of 1·282 leads to acceptance of the alternative hypothesis, namely that students from college P take longer than those from college Q. Since the question states explicitly that there is no evidence that the population variances are equal, it is inappropriate to base the test on a pooled estimate of common variance, as a surprisingly large proportion of candidates did. Most candidates knew how to find a confidence interval for the difference in the population means, here requiring a tabular value of 1·645, and which may be stated in alternative ways.

Answer: **(ii)** [0·09, 2·71].

Question 11 (Mechanics)

This optional question was attempted by a minority of candidates. Many of them were able to correctly formulate two equations relating the tension and speed at any point by using conservation of energy and a radial resolution of forces, and then combine them to give an expression for the tension. Equating this to the given value at which the string breaks verifies the given value of $\cos\theta$, and substituting this into the previously found energy equation yields the required speed v. The second part proved to be more challenging. By considering the subsequent horizontal motion after the string breaks, the time which elapses until P is vertically below O is found to be $\frac{4a}{3v}$, while consideration of vertical motion yields the height fallen and hence the required distance OP.

Answers: **(i)** $\sqrt{\frac{8ag}{5}}$; **(ii)** $\frac{20a}{9}$.

Question 11 (Statistics)

The required value of p is first found by finding an expression in terms of p for the gradient of the regression line of y on x and equating this expression to the given gradient 0·25. The required value of k then follows from recalling that the mean values of x and y satisfy the equation of the regression line. Candidates should be aware that, while the sample means satisfy the equation of a regression line, a pair of sample values do not necessarily do so. The required value of the correlation coefficient is found using the standard formula. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho > 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison of the coefficient value found earlier with the tabular value 0·729 leads to a conclusion of there being no evidence of positive correlation.

Answers: **(i)** 6·5, 5; **(ii)** 0·659.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a suitable diagram. Annotating a diagram printed in the question paper may be sufficient if the result is clear, but it may well be preferable to draw a fresh diagram within the answer.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Question 5** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 4** and the directions of motion of particles. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 7, 10 and 11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

Comments on specific questions

Question 1

Although various approaches to finding the mass *m* of the bullet are possible, a straightforward one is to equate the change in momentum, $m(250 - 70)$, to the impulse causing it, $450 \times 0{\cdot}04$. Another is to find the deceleration of the bullet and apply Newton's second law of motion. Many candidates made a good attempt, but care must be taken over the signs, to avoid the implicit assumption that the bullet reverses direction, for example.

Answer: 0·1.

Question 2

Since *P* is moving in simple harmonic motion, equating the ratio of its speeds at *A* and *B* to 3 : 4 using the standard result $v^2 = \omega^2(a^2 - x^2)$ leads to an equation for a^2, and hence the required value of the amplitude *a*. The second part similarly requires the use of standard SHM results, namely maximum speed = ωa to give $\omega = \frac{\pi}{6}$ and then $T = \frac{2\pi}{w}$ to give the period *T*. The required time from *A* to *B* may be found in different ways, but in all cases it is important to consider whether the expression $a \sin \omega t$ or $a \cos \omega t$ is the appropriate one for each of the distances being found.

Answers: **(i)** 2 m; **(ii)** 12 s; **(iii)** 3·00 s.

Question 3

Many candidates were able to formulate two equations for the speed v_A of the sphere *A* after the collision by means of conservation of momentum and Newton's restitution equation, and then eliminate v_A to verify the given expression for *e*. It was clear that some candidates needed to go back and correct a sign or signs in their initial working in order to achieve the given result. Equating the final kinetic energy $\frac{1}{2}mv_A^2$ to the appropriate proportion of the initial kinetic energy $\frac{1}{2}m(1 + k^2)u^2$ leads to a quadratic equation in *k* and hence the required value of *e*. The proportion to be used in this approach is 0·4, but 0·6 was quite often seen. The quadratic equation has of course two roots, and candidates should take care to select the appropriate one, which here means rejecting $k = \frac{1}{3}$ and hence the unacceptable value $-\frac{1}{2}$ for *e*.

Answer: **(ii)** 3, $\frac{1}{2}$.

Question 4

As in all similar questions, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram so that the symbols used to represent them are clear. Following the instruction in the question, most candidates succeeded in verifying the given magnitude of the normal reaction at *B* by taking moments about *A*. A similar process gives the new magnitude after the movement of the particle up the rod, and using the given information about the ratio of the two magnitudes verifies the given value of *x*. Finally the magnitudes of the frictional and normal reaction forces at *A* are found by horizontal and vertical resolution of the forces on the rod when the particle is in its second position, and equating their ratio to the given coefficient of friction gives the required value of $\tan \theta$. Taking moments about *B*, say, is an alternative to a horizontal resolution, but with the disadvantage of more working.

Answers: **(iii)** 1·25.

Question 5

Verifying the given moment of inertia I requires the use of standard formulae and both the perpendicular and parallel axis theorems. The obvious, and popular, approach is to formulate and then sum the individual moments of inertia of the rings about the axis l, in which case it is of course necessary to first apply the perpendicular axis theorem, and then the parallel axis theorem, since the reverse order gives an incorrect result. An alternative is to find the moments of inertia of the three rings both about the line of symmetry of the object (intersecting the axis l at O, say) and about an axis through O which is perpendicular to the plane of the rings. The perpendicular axis theorem may then be applied at O. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms for I with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. Candidates who initially fail to achieve the given result, whether through not making use of the perpendicular axis theorem, misusing the parallel axis theorem to shift the axis of rotation from a point which is not the centre of mass, or finding a distance incorrectly, should look for their error instead of adding or subtracting a spurious term whose only justification is that it produces the given result, since this is unlikely to satisfy the Examiners. In the second part, the couple acting on the system should be found in terms of sin θ, where θ is the small angular displacement, and equated to $I\ \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$. Approximating sin θ by θ yields the familiar form $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\theta$ of the standard SHM equation, in which the minus sign is essential, and from which the required period $\frac{2\pi}{w}$ is obtained. While this final result may be stated in a number of acceptable forms, there should be some attempt at simplification. Candidates should be very much aware that the constant parameter ω in the SHM equation is not equal to $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ even though the same symbol is sometimes used to denote the latter, and thus finding $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ from conservation of energy at some arbitrary angular displacement and equating it to the SHM parameter ω is wholly invalid.

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Most candidates produced a correct answer to the first part, by finding 1 – F(2). The meaning of interquartile range was not always well understood, and even some candidates who found the values of the upper and lower quartiles correctly, namely 3·466 and 0·719, did not appreciate that the interquartile range means their difference.

Answers: **(i)** 0·449; **(ii)** 2·75.

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Answers: **(i)** $\sqrt{\frac{8ag}{5}}$; **(ii)** $\frac{20a}{9}$.

Question 11 (Statistics)

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Answers: **(i)** 6·5, 5; **(ii)** 0·659.

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

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Comments on specific questions

Question 1

Most candidates found this question straightforward, though a few confused radial and transverse. The required radial component of *P*'s acceleration is found using the standard formula $\frac{v^2}{r}$, though having written down this formula correctly, and found v to be 4 when $t = 2$, candidates must not forget to square their value of v before dividing by 0·8. The tangential component follows from differentiation of the given expression for the velocity v, again setting $t = 2$. There is no need to introduce angular speed or acceleration in either part.

Answer: 20 m s^{-2}; 3 m s^{-2} .

Question 2

Two equations for the speeds of the spheres *A* and *B* after their collision were usually formulated correctly by means of conservation of momentum and Newton's restitution equation, with the given result for *A*'s speed providing a useful check on their solution. The speed of *B* after its collision with the barrier follows from a further application of Newton's restitution equation, and equating the magnitude of the resulting expression to that for the speed of *A* yields a quadratic equation for *e*. This has of course two roots, and candidates should take care to select the appropriate one, which here means rejecting the value –2 for *e*. While most candidates stated correctly that sphere *B* does collide with the barrier for a second time, their method for determining this was often inadequate or erroneous. A straightforward approach is to note that prior to the second collision between *A* and *B*, the two particles are moving with the same speed $\frac{2}{3}u$ but in opposite directions. Using once again conservation of momentum and Newton's restitution equation shows that after the collision *B* is moving towards the barrier (with speed $10u/9$ or an equivalent expression) and thus collides with it again. It is perhaps simpler to note that the total momentum of *A* and *B* before their second collision is directed towards the barrier (namely $4mv_A – mv_A = 2mu$), so *B* will necessarily move towards the barrier after the collision.

Answers: **(i)** $\frac{4}{5}u\,(1 + e)$; **(ii)** $\frac{2}{3}$.

Question 3

Since *P* is moving in simple harmonic motion, equating the magnitude of its acceleration at *B* to twice that at *A* leads to $7·5 – d = 2\,(6·5 – d)$, and hence the required value of *d*. The second part similarly requires the use of standard SHM results, namely $T = \frac{2\pi}{w}$ to give $\omega = 2$, maximum acceleration = $\omega^2 a$ to give the amplitude a = 2·5 and finally $v^2 = \omega^2\,(a^2 – x^2)$ to give the required speed v when $x = 7 – d = 1·5$. The required time from *A* to *B* may be found in different ways, but in all cases it is important to consider whether the expression $a \sin \omega t$ or $a \cos \omega t$ is the appropriate one for the distance being found.

Answers: **(i)** 5·5; **(ii)** 4 m s^{-1}; **(iii)** 0·258 s.

Question 4

Verifying the given length of *AC* was usually done correctly, most often by combining $AD = 2a \cos \theta$ and $AD = \frac{AC}{\cos\theta}$. In the second part, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram so that the symbols used to represent them are clear. This process reveals that four possible forces are unknown, namely the normal reaction R_A and friction F_A at *A*, the normal reaction R_B at *B*, and the tension *T* in the rope, implying that four independent equations will be needed to verify the given value of *T*. One such follows immediately from the information about the forces at *A* in the question, namely $F_A = \frac{R_A}{4}$. While there are a variety of possible moment and force resolution equations to choose from, many candidates chose to resolve forces vertically and horizontally and to take moments about some point, most often *A* or *B*. As is sometimes the case in such questions, it is possible to reduce the work required by a careful choice of equations. Thus taking moments about B, for example, together with only a vertical resolution of forces is sufficient to give T since the unknown force R_B is not introduced here (though it is required in the final part of the question). Knowing T, the magnitudes of R_A and R_B required in part **(iii)** are readily found from the appropriate moment and/or resolution equations.

Answers: **(iii)** $\frac{23W}{20}$, $\frac{39W}{80}$.

Question 5

It is first necessary to find the speeds (or the square of the speeds) of P immediately before and after its collision with Q at the lowest point. That before follows from conservation of energy and that after from conservation of momentum (but not energy, as some candidates wrongly believed). The corresponding tensions may be found by using the general formula $\frac{mv^2}{a} + mg$ with appropriate values of v and m, and in connection with the tension after the collision it is important to recall that the two particles have then coalesced. Equating half the tension before the collision to the tension after the collision gives a quadratic equation for k, with two acceptable solutions. A complete response to the question thus requires several steps, and candidates are less likely to become confused by their various equations if they preface each step by a very brief descriptor, such as ‘by conservation of momentum’.

Answers: 2·5, 3.

Question 6

Most candidates stated the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho \neq 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison with the tabular value 0·441 leads to a conclusion of there being no evidence of non-zero correlation. The second part requires a comparison of the given value 0·615 with tabulated one-tail 5% critical values of the product moment correlation coefficient, with sufficient explanation to justify the chosen answer. This could involve noting that 0·615 lies between the adjacent tabular values 0·621 and 0·582.

Answers: **(ii)** 9

Question 7

The given quadratic equation may be verified by equating the given variance of X to $\frac{(1-p)}{p^2}$, and when solving it candidates should reject the inadmissible value $-\frac{2}{3}$ of p. $P(X = 5)$ was usually found correctly from $(1 - p)^4 p$, using the value of p found earlier, while $P(3 \leqslant X \leqslant 7)$ is most easily found from $(1 - p)^2 - (1 - p)^7$. Candidates should note that this latter probability is not equal to $P(X \leqslant 7) - P(X \leqslant 3)$.

Answers: **(i)** 0·4; **(ii)** 0·0518; **(iii)** 0·332.

Question 8

The required value of the gradient b of the regression line of y on x follows from the fact that the square of the product moment correlation coefficient equals the product of the gradients of the two regression lines, as most candidates appreciated (though some overlooked the square). The key to the second part is that the means of x and y satisfy the regression line of y on x, giving the mean 4·251 of x and hence the required value of the unknown sample value p. The same relationship between the means and the regression line of x on y yields the value –0·6560 of d and hence the required value of x when $y = 8·5$. Since the exact value is only very slightly greater than 4·725, only the retention of additional figures in the intermediate results can determine whether the value rounded to 3 significant figures should be 4·72 or 4·73. Indeed, following the spirit rather than the letter of the accuracy rubric, it may be sensible to give this answer to 4 rather than just 3 significant figures.

Answers: **(i)** 1·516; **(ii)** 4·20; **(iii)** 4·73.

Question 9

Many candidates found the distribution function F(x) over $1 \leqslant x \leqslant 9$ correctly by integrating f(x). The next step is to find or state the distribution function G of Y over $1 \leqslant y \leqslant 3$, namely $\frac{(3y^2 - 2y - 1)}{20}$, which is then differentiated to verify the given probability density function g(y). Although the constant of integration $\frac{1}{20}$ in F(x) is strictly irrelevant to g(y), candidates should not simply omit this constant without explaining why they are justified in doing so. The method for finding the required mean value of Y was generally known, namely integration of y g(y), though the mean was sometimes confused with the median.

Answers: **(ii)** $\frac{11}{5}$

Question 10

The question does not explicitly specify the use of a particular test in the first part, but most candidates realised that a paired-sample t-test is appropriate, and conducted the test well. Use of some other test is not acceptable. The first step in the calculation is to find the differences between the pairs of observations so that the test may be based on them. It is usually essential to retain the signs of the differences and not just consider their magnitudes, but as it happens they are all here of the same sign. The mean of the resulting sample is then of magnitude 1·4 and the unbiased estimate of the population variance is 0·76. This gives a calculated value of t of magnitude 1·38, and comparison with the tabulated value 1·397 leads to the conclusion of there being no evidence to support the coach's belief. As in all such tests, candidates should state their hypotheses explicitly in terms of the population rather than the sample means, and preferably in appropriate mathematical notation rather than in words. The unbiased estimate of the population variance needed in the second part is that for y alone, and is found to be 6·693. Use of this estimate and the critical t-value 2·306 then gives the required confidence interval, expressed in any equivalent form.

Answers: **(ii)** [33·5, 37·5].

Question 11 (Mechanics)

This optional question was attempted by fewer than half the candidates, but many of those who did so made good attempts, particularly at the first part. This requires the use of standard formulae and the parallel axes theorem to formulate and then sum the individual moments of inertia of the components of the object. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Those who simply write down a sum of several terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. In the second part of the question many candidates realised that the gain in rotational energy of the object should be equated to its loss in potential energy when D is vertically below A, though some found calculation of the latter difficult. Solution of the resulting equation with the given angular speed of the object inserted yields the required value of k.

Answers: **(ii)** 4.

Question 11 (Statistics)

Candidates largely knew how to calculate estimates of the mean and variance, though in the latter case some overlooked the requirement for an unbiased estimate. Not all knew that the mean and variance of a binomial distribution B(n, p) are np and $np(1 – p)$ respectively. The similarity of the corresponding values found for X and B(6, 0·313) are an acceptable explanation. In the second part, it is sufficient to state $250 \times {}^6C_4 \times 0{\cdot}313^4 \times 0{\cdot}687^2$, for example, and most candidates did so. In the final part a clear statement of the null hypothesis, such as 'the binomial distribution is a good fit to the data', is preferable to a more vague statement such as 'it fits'. Most candidates were apparently aware that in such questions cells may need to be combined in order to ensure that all the expected values are at least 5, and here the last three cells must be so combined. Indeed the goodness of fit test was often carried out well. Candidates should take particular care over the number of degrees of freedom when quoting the appropriate critical value. Because the question implies that the scientist has determined the parameter 0·313 from the observational data, 5 cells correspond to 3 degrees of freedom here, giving a critical value 7·815. A comparison with the calculated value 3·81 of X^2 leads to acceptance of the null hypothesis, and hence the conclusion that the scientist's belief is correct.

Answers: **(i)** 1·878, 1·290.