

Example Candidate Responses

Cambridge
International
AS & A Level

Cambridge International AS and A Level Mathematics

9709

Paper 3

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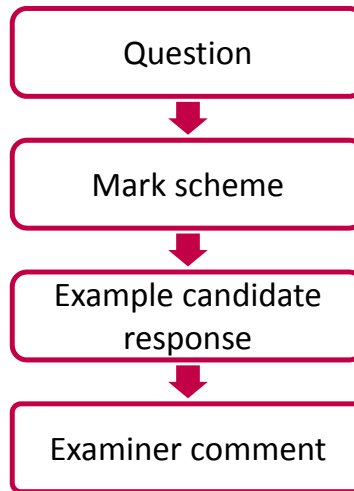
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Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Mathematics (9709), and to show how different levels of candidates' performance relate to the subject's curriculum and assessment objectives.

In this booklet candidate responses have been chosen to exemplify a range of answers. Each response is accompanied by a brief commentary explaining the strengths and weaknesses of the answers.

For ease of reference the following format for each component has been adopted:



Each question is followed by an extract of the mark scheme used by examiners. This, in turn, is followed by examples of marked candidate responses, each with an examiner comment on performance. Comments are given to indicate where and why marks were awarded, and how additional marks could have been obtained. In this way, it is possible to understand what candidates have done to gain their marks and what they still have to do to improve them.

Past papers, Examiner Reports and other teacher support materials are available on Teacher Support at <https://teachers.cie.org.uk>

Assessment at a glance

The 7 units in the scheme cover the following subject areas:

- Pure Mathematics (units P1, P2 and P3);
- Mechanics (units M1 and M2);
- Probability and Statistics (units S1 and S2).

Centres and candidates may:

- take all four Advanced (A) Level components in the same examination session for the full A Level.
- follow a staged assessment route to the A Level by taking two Advanced Subsidiary (AS) papers (P1 & M1 **or** P1 & S1) in an earlier examination session;
- take the Advanced Subsidiary (AS) qualification only.

AS Level candidates take:

Paper 1: Pure Mathematics 1 (P1)
<p>1¾ hours About 10 shorter and longer questions 75 marks weighted at 60% of total</p>

plus **one** of the following papers:

Paper 2: Pure Mathematics 2 (P2)	Paper 4: Mechanics 1 (M1)	Paper 6: Probability and Statistics (S1)
<p>1¾ hours About 7 shorter and longer questions 50 marks weighted at 40% of total</p>	<p>1¾ hours About 7 shorter and longer questions 50 marks weighted at 40% of total</p>	<p>1¾ hours About 7 shorter and longer questions 50 marks weighted at 40% of total</p>

Assessment at a glance

A Level candidates take:

Paper 1: Pure Mathematics 1 (P1)	Paper 3 Pure Mathematics 3 (P3)
1¼ hours About 10 shorter and longer questions 75 marks weighted at 30% of total	1¼ hours About 10 shorter and longer questions 75 marks weighted at 30% of total

plus **one** of the following combinations of two papers:

Paper 4: Mechanics 1 (M1)	Paper 6: Probability and Statistics 1 (S1)
1¼ hours About 7 shorter and longer questions 50 marks weighted at 20% of total	1¼ hours About 7 shorter and longer questions 50 marks weighted at 20% of total

OR

Paper 4: Mechanics 1 (M1)	Paper 5: Mechanics 2 (M2)
1¼ hours About 7 shorter and longer questions 50 marks weighted at 20% of total	1¼ hours About 7 shorter and longer questions 50 marks weighted at 20% of total

OR

Paper 6: Probability and Statistics 1 (S1)	Paper 7: Probability and Statistics 2 (S2)
1¼ hours About 7 shorter and longer questions 50 marks weighted at 20% of total	1¼ hours About 7 shorter and longer questions 50 marks weighted at 20% of total

Teachers are reminded that the latest syllabus is available on our public website at www.cie.org.uk and Teacher Support at <https://teachers.cie.org.uk>

Paper 3

Question 1

- 1 Solve the equation

$$\ln(x+5) = 1 + \ln x,$$

giving your answer in terms of e.

[3]

Mark scheme

- 1 State or imply $\ln e = 1$ B1
 Apply at least one logarithm law for product or quotient correctly
 (or exponential equivalent) M1
 Obtain $x+5 = ex$ or equivalent and hence $\frac{5}{e-1}$ A1 [3]

Example candidate response – 1

1.	$\ln(x+5) = 1 + \ln x$	
	$\ln(x+5) - \ln x = 1$	
	$\ln\left(\frac{x+5}{x}\right) = 1$ ✓	
	$\frac{x+5}{x} = e^1$ ✓	M1
		B1
	$x = xe^1 - 5$ // A	A0
		(2)

Total mark awarded = 2 out of 3

Example candidate response – 2

$\ln(x+5) = 1 + \ln x$		
$\ln(x+5) - \ln x = 1$		
log $\ln(x+5)x \neq 1$	M	0
$(x+5)x = e^1$	B	1
$x^2 + 5x =$		
$\therefore x = e^1 \quad x = e^1 - 5$	A	0
		(1)

Total mark awarded = 1 out of 3

Examiner comments

This question required knowledge of the rules for logarithms and exponentials. The question also made a specific request for an expression for x in terms of e .

Candidate 2 demonstrated some understanding of the rules of logarithms, but was confused between products and quotients, resulting in $(x+5)x$ instead of $\frac{x+5}{x}$. Although it was quite common to see errors in changing from logarithms to exponential form, this particular candidate had completed that step correctly, although they then proceeded to reach inappropriate conclusions about the solutions of their equation.

Candidate 1 reached a correct expression in x and e , but did not then rearrange this to find x in terms of e .

Question 2

- 2 (i) Express $24 \sin \theta - 7 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the value of α correct to 2 decimal places. [3]

- (ii) Hence find the smallest positive value of θ satisfying the equation

$$24 \sin \theta - 7 \cos \theta = 17. \quad [2]$$

Mark scheme

- 2 (i) State or imply $R=25$ B1
 Use correct trigonometric formula to find α M1
 Obtain 16.26° **with no errors seen** A1 [3]
- (ii) Evaluate of $\sin^{-1} \frac{17}{R}$ ($= 42.84\dots^\circ$) M1
 Obtain answer 59.1° A1 [2]

Example candidate response – 1

	2) i)	$24 \sin \theta - 7 \cos \theta = R \sin(\theta - \alpha)$
		$= R \sin \theta \cos \alpha - R \sin \alpha \cos \theta$
	By comparing,	
		$R \cos \alpha = 24 \quad \text{--- ①}$
		$24 \cos \alpha$ $- R \sin \alpha = -7$
		$R \sin \alpha = 7 \quad \text{--- ②}$
	② ①	
M1		$\tan \alpha = \frac{7}{24}$
A0		$d = 16.3^\circ$ $24 \sin \theta - 7 \cos \theta = 25 \sin(\theta - 16.3^\circ)$ *
B1		$R \sin 16.3 = 7$ $R = 25$ ✓

	2) ii)	$24 \sin \theta - 7 \cos \theta = 17$	For Examiner's use only
		$25 \sin(\theta - 16.3^\circ) = 17$	
		$\theta - 16.3^\circ = \sin^{-1}\left(\frac{17}{25}\right)$	
		$\theta - 16.3^\circ = 42.8^\circ$	
		$\theta = 59.1^\circ$ *	
		/	
			2
			④

Item marks awarded: (i) = 2/3; (ii) = 2/2

Total mark awarded = 4 out of 5

Example candidate response – 2

2. (i) ~~R = 25~~ (2)

$$24 \sin \theta - 7 \cos \theta \equiv R \sin(\theta - \alpha)$$

$$\alpha \Rightarrow \tan \alpha = \frac{24}{7}$$

$$\alpha = \tan^{-1}\left(\frac{24}{7}\right)$$

$$\alpha = 73.73 \dots$$

$$\alpha = 73.7$$

$$R \Rightarrow = \frac{24}{25} \sqrt{24^2 + (-7)^2}$$

$$= 25$$

$\alpha = \tan^{-1}\left(\frac{24}{7}\right)$ M0

$$\alpha = 73.73 \dots$$

$$\alpha = 73.7$$

$$\therefore 24 \sin \theta - 7 \cos \theta = 25 \sin(\theta - 73.7)$$

(ii) $25 \sin(\theta - 73.7) = 17$ SIA

$$\sin(\theta - 73.7) = \frac{17}{25}$$

$$\frac{S}{T} = \frac{A}{C}$$

$$\theta - 73.7 = \sin^{-1}\left(\frac{17}{25}\right)$$

$$\sin \alpha = \frac{17}{25}$$

$$\alpha = \sin^{-1}\left(\frac{17}{25}\right)$$

$$\alpha = 42.84 \dots$$

$$\alpha = 42.8$$

$$\theta - 73.7 = 42.8, 137.2$$

$$\theta = 42.8 + 73.7, 137.2 + 73.7$$

$$\theta = 116.5^\circ, 210.9^\circ$$

$\alpha = 42.8$ M1

$\theta = 116.5^\circ, 210.9^\circ$ A0

(2)

Item marks awarded: (i) = 1/3; (ii) = 1/2

Total mark awarded = 2 out of 5

Examiner comments

- (i) The majority of candidates approached the task of expressing $24 \sin \theta - 7 \cos \theta$ in the form $R \sin(\theta - \alpha)$ by quoting formulae they had memorised for R and α . This frequently resulted in incorrect statements, as seen here in the work of candidate 2. The confusion between θ and α in the first line of this candidate's response could have been overlooked, but quoting the negative reciprocal of the required value for $\tan \alpha$ resulted in not being awarded the method mark. If they had started with a correct expansion of $R \sin(\theta - \alpha)$ and had compared coefficients before making their error, the method mark would have been available. The working for R is independent of the work to find α , so this candidate does score the mark for a correct value of R . Candidate 1 has shown the expansion of $R \sin(\theta - \alpha)$, together with explanations of their methods for finding R and α , but the question asked for the value of α to two decimal places, and in this response we do not see anything more accurate than 16.3° , so the accuracy mark was not awarded.
- (ii) Candidate 2 has gone on to use their answer from part (i) correctly, but their previous error means that they cannot obtain the correct answer here. Candidate 1 is fortunate that they have sufficient accuracy in their value for α to obtain the correct value for θ .

Question 3

3 The parametric equations of a curve are

$$x = \frac{4t}{2t+3}, \quad y = 2 \ln(2t+3).$$

(i) Express $\frac{dy}{dx}$ in terms of t , simplifying your answer. [4]

(ii) Find the gradient of the curve at the point for which $x = 1$. [2]

Mark scheme

3 (i) Either Use correct quotient rule or equivalent to obtain

$$\frac{dx}{dt} = \frac{4(2t+3) - 8t}{(2t+3)^2} \text{ or equivalent} \quad \text{B1}$$

$$\text{Obtain } \frac{dy}{dt} = \frac{4}{2t+3} \text{ or equivalent} \quad \text{B1}$$

$$\text{Use } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ or equivalent} \quad \text{M1}$$

$$\text{Obtain } \frac{1}{3}(2t+3) \text{ or similarly simplified equivalent} \quad \text{A1}$$

Or Express t in terms of x or y e.g. $t = \frac{3x}{4-2x}$ B1

$$\text{Obtain Cartesian equation e.g. } y = 2 \ln\left(\frac{6}{2-x}\right) \quad \text{B1}$$

$$\text{Differentiate and obtain } \frac{dy}{dx} = \frac{2}{2-x} \quad \text{M1}$$

$$\text{Obtain } \frac{1}{3}(2t+3) \text{ or similarly simplified equivalent} \quad \text{A1} \quad [4]$$

(ii) Obtain $2t=3$ or $t=\frac{3}{2}$ B1

Substitute in expression for $\frac{dy}{dx}$ and obtain 2 B1 [2]

Example candidate response – 1

3. (i) $x = \frac{4t}{2t+3}$ u $y = 2 \ln(2t+3)$
 v $\frac{dy}{dx} = 2 \left(\frac{2}{2t+3} \right)$

$u = 4t$ $v = 2t+3$
 $\frac{du}{dt} = 4$ $\frac{dv}{dt} = 2$ $= \frac{6}{2t+3}$

$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$= \frac{(2t+3)(4) - (4t)(2)}{(2t+3)^2}$

$= \frac{(8t+12) - 4t}{(2t+3)^2}$

$= \frac{4t+12}{(2t+3)^2}$

$= \frac{4t+18}{(2t+3)^2}$

$\frac{dy}{dx} = \frac{6}{2t+3} \times \frac{4t+18}{(2t+3)^2}$

$= \frac{6(4t+18)}{(2t+3)^3} = \frac{12t+18}{(2t+3)^3}$

(ii) At $x=1$ $At = m \frac{3}{2}$
 $1 = \frac{4t}{2t+3}$ $\frac{dy}{dx} = \frac{12 \left(\frac{m \frac{3}{2} \right) + 18}{4 \left(\frac{m \frac{3}{2} \right) + 18}$
 $2t+3 = 4t$ $= \frac{3}{2}$
 $+3 = 4t - 2t$
 $+3 = 2t$
 $t = \frac{3}{2}$

(3)

Item marks awarded: (i) = 2/4; (ii) = 1/2

Total mark awarded = 3 out of 6

Example candidate response – 2

Question 3

(i) $x = \frac{4t}{2t+3}$ $u \rightarrow u' = 4$ $v \rightarrow v' = 2$

$y = 2 \ln(2t+3)$

$\frac{dy}{dt} = 2 \left(\frac{1}{2t+3} \right)$

$\frac{dx}{dt} = uv' + vu'$

$\frac{dy}{dt} = \frac{2}{2t+3}$

$= 4t(2) + (2t+3)4$

$= 8t + 8t + 12$

$= 16t + 12$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2}{2t+3} \times \frac{1}{16t+12}$

$\frac{(2t+3)(16t+12)}{32t^2 + 24t + 48t + 36}$

$\frac{32t^2 + 72t + 36}{4(8t^2 + 18t + 9)}$

$= \frac{2}{4} \times \frac{1}{(8t^2 + 18t + 9)}$

$= \frac{1}{2(8t^2 + 18t + 9)}$ (ans)

(ii) Grad = 1 when $x=1$

Grad = $\frac{1}{16t+12}$

$\rightarrow \frac{1}{70}$

BO

BO

(1)

Item marks awarded: (i) = 1/4; (ii) = 0/2

Total mark awarded = 1 out of 6

Examiner comments

- (i) This part of the question required the candidates to start by differentiating a quotient and a logarithmic function. Candidate 2 started by trying to use the product rule, but did not express the function as a product, causing their method to fail. In differentiating the logarithmic function, they have lost a factor of 2. Despite these two errors, they have used the chain rule correctly to obtain an expression for $\frac{dy}{dx}$ in terms of t , and so scored a method mark.

Candidate 1 has quoted the quotient rule correctly, but has made an error in applying it; the numerator of their expression has the term $(2t + 3)t$ where it should have $4t \times 2$. Although they have called it $\frac{dy}{dx}$,

they do have a correct unsimplified expression for $\frac{dy}{dt}$. They have scored a mark for the correct unsimplified form, despite the subsequent error in simplifying their answer. They then go on to use their two derivatives correctly, so they too score the method mark for using the chain rule to obtain an expression for $\frac{dy}{dx}$.

- (ii) This part of the question required the candidate to use their answer from part (i) to find the gradient when $x = 1$, so the first step is to find the value of t when $x = 1$. In common with many other candidates, candidate 2 has used the value for x as the value for t , so they do not earn any marks here.

Candidate 1 does find the correct value for t , and substitutes it in to their expression for the gradient, but their earlier error means that they cannot obtain the correct final answer here.

Question 4

- 4 The variables x and y are related by the differential equation

$$(x^2 + 4) \frac{dy}{dx} = 6xy.$$

It is given that $y = 32$ when $x = 0$. Find an expression for y in terms of x .

[6]

Mark scheme

- | | | | |
|---|--|----|-----|
| 4 | Separate variables correctly and integrate one side | M1 | |
| | Obtain $\ln y = \dots$ or equivalent | A1 | |
| | Obtain $= 3 \ln(x^2 + 4)$ or equivalent | A1 | |
| | Evaluate a constant or use $x = 0, y = 32$ as limits in a solution containing terms $a \ln y$ and $b \ln(x^2 + 4)$ | M1 | |
| | Obtain $\ln y = 3 \ln(x^2 + 4) + \ln 32 - 3 \ln 4$ or equivalent | A1 | |
| | Obtain $y = \frac{1}{2}(x^2 + 4)$ or equivalent | A1 | [6] |

Example candidate response – 1

QUESTION 4.

$$(x^2 + 4) \frac{dy}{dx} = 6xy$$

$$\int \frac{1}{y} dy = \int \frac{6x}{(x^2+4)} dx$$

$$\ln y = \int \frac{6}{(x^2+4)} dx$$

$$\ln y = 3 \int \frac{2x}{(x^2+4)} dx$$

$$\ln y = 3 \ln(x^2+4) + C$$

$$y = 32, x = 0$$

$$\ln 32 = 3 \ln(4) + C$$

$$\ln 32 = \ln 64 + C$$

$$C = \frac{\ln 32 - \ln 64}{1} = \frac{\ln \frac{32}{64}}{1}$$

$$C = \ln \frac{1}{2}$$

$$\ln y = 3 \ln(x^2+4) + \frac{1}{2}$$

$$y = 3(x^2+4) + e^{\frac{1}{2}}$$

$$y = 3x^2 + 12 + e^{\frac{1}{2}}$$

$$\frac{6x}{(x^2+2)(x+2)} = \frac{A}{x+2} + \frac{B}{x+2}$$

let $x = -2$

$$-12 = -6 = A + B$$

$$A = -6 - B$$

let $x = 0$

$$0 = 2(-6 - B) + 2B$$

$$0 = -12 - 2B$$

3

M

10

AO

(4)

Total mark awarded = 4 out of 6

Example candidate response – 2

$(x^2 + 4) \frac{dy}{dx} = 6xy$	
$\int \frac{1}{y} dy = \int \frac{6x}{(x^2+4)} dx$	
$\int \frac{6x}{x^2+4} dx$	
$u = 6x \quad dv = (x^2+4)^{-1}$	
$\frac{du}{dx} = 6$	
$v = \frac{1}{2x} \ln(x^2+4)$	M
$\ln y + c = \left(\frac{6x \cdot \ln(x^2+4)}{2x} \right) - \int \frac{6 \ln(x^2+4)}{2x} dx$	
$\ln y + c = \left(\frac{6x \cdot \ln(x^2+4)}{2x} \right) - \int \frac{6 \ln(x^2+4)}{2x} dx$	
$\ln y + c = \frac{3x \ln(x^2+4)}{2} - \int \frac{6 \ln(x^2+4)}{2x} dx$	A
	A0
	(2)

Total mark awarded = 2 out of 6

Examiner comments

The essential first step in solving this differential equation is to separate the variables and integrate both sides of the resulting equation. In both of these examples the separation of the variables has been completed successfully. Despite an initial (incorrect) thought that they should try to use partial fractions to split up the fraction, candidate 1 does complete both integrals correctly. They then substitute values correctly to find an expression for the constant of integration, however, their constant becomes $\frac{1}{2}$ when it should be $\ln \frac{1}{2}$, so they do not reach a correct expression for $\ln y$ and cannot reach the correct final answer.

Candidate 2's work suggests that they do know that $\frac{d}{dx}(\ln(x^2 + 4)) = \frac{2x}{x^2 + 4}$ but they have not realised that

this means that they can simply write down an expression for $\int \frac{6x}{x^2 + 4} dx$. They have made an incorrect (and incomplete) attempt to use integration by parts. They have stopped work at this point but, as their expression does not contain terms in x and y of the correct form, no further marks were available to them.

Question 5

5 The expression $f(x)$ is defined by $f(x) = 3xe^{-2x}$.

(i) Find the exact value of $f'(-\frac{1}{2})$. [3]

(ii) Find the exact value of $\int_{-\frac{1}{2}}^0 f(x) dx$. [5]

Mark scheme

- 5 (i) Either Use correct product rule M1
 Obtain $3e^{-2x} - 6xe^{-2x}$ or equivalent A1
 Substitute $-\frac{1}{2}$ and obtain $6e$ A1
- Or Take ln of both sides and use implicit differentiation correctly M1
 Obtain $\frac{dy}{dx} = y \left(\frac{1}{x} - 2 \right)$ or equivalent A1
 Substitute $-\frac{1}{2}$ and obtain $6e$ A1 [3]
- (ii) Use integration by parts to reach $kxe^{-2x} \pm \int ke^{-2x} dx$ M1
 Obtain $-\frac{3}{2}xe^{-2x} + \int \frac{3}{2}e^{-2x} dx$ or equivalent A1
 Obtain $-\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x}$ or equivalent A1
 Substitute correct limits correctly DM1
 Obtain $-\frac{3}{4}$ with no errors or inexact work seen A1 [5]

Example candidate response – 1

5.	$f(x) = \underbrace{3x}_u \underbrace{e^{-2x}}_v$	$u = 3x$	$v = e^{-2x}$	
(i)		$u' = 3$	$v' = -2e^{-2x}$	
	$f'(-\frac{1}{2}) = e^{-2(-\frac{1}{2})} \cdot 3 + \cancel{3(-\frac{1}{2})} \cdot -2e^{-2(-\frac{1}{2})}$			M 1
	$= 3e^1 + \cancel{6} \cdot \frac{3}{2} \cdot \cancel{-2}e^1$			A 1
	$\therefore f'(-\frac{1}{2}) = 3e^1 + 3e^1$	//	A	
	$\therefore f'(-\frac{1}{2}) = 16.3$	#		A 0
(ii)	$\int_{-\frac{1}{2}}^0 \underbrace{3x}_u \underbrace{e^{-2x}}_{v'} dx$	$u = 3x$	$v = -\frac{e^{-2x}}{2}$	
		$u' = 3$	$v' = e^{-2x}$	
	$= \left[3x \cdot -\frac{e^{-2x}}{2} \right]_{-\frac{1}{2}}^0 - \int_{-\frac{1}{2}}^0 -\frac{e^{-2x}}{2} \cdot 3 dx$			
	$= \left[\frac{-3xe^{-2x}}{2} \right]_{-\frac{1}{2}}^0 - \left[\cancel{\frac{-3e^{-2x}}{2}} - \frac{-3e^{-2x}}{2 \times -2} \right]_{-\frac{1}{2}}^0$			3
	$= \left[\frac{-3(0)e^{-2(0)}}{2} - \left[\frac{-3(-\frac{1}{2})e^{-2(-\frac{1}{2})}}{2} \right] \right] - \left[\frac{-3e^{-2(0)}}{-4} - \left(\frac{-3e^{-2(-\frac{1}{2})}}{-4} \right) \right]$			
	$= 0 - \cancel{24} \frac{3e^1}{4} - \left[\frac{3}{4} - \frac{3}{4}e^1 \right]$			
	$= \frac{3e^1}{4} + \frac{3e^1}{4} - \frac{3}{4}$			M 1
	$= \frac{3}{2}e^1 - \frac{3}{4}$	//		A 0
				(6)

Item marks awarded: (i) = 2/3; (ii) = 4/5

Total mark awarded = 6 out of 8

Example candidate response – 2

Question 5

5. i) $f(x) = 3x e^{-2x}$

$$u = 3x \quad \left\{ \begin{array}{l} v = e^{-2x} \\ \frac{du}{dx} = 3 \\ \frac{dv}{dx} = -2e^{-2x} \end{array} \right.$$

$$f'(x) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$= 3x \times (-2e^{-2x}) + e^{-2x} \times 3$$

$$= -3x e^{-2x} + 3e^{-2x}$$

$$= 3e^{-2x} (1-x)$$

$f'(-\frac{1}{2}) = 3e^{-2(-\frac{1}{2})} (1 - (-\frac{1}{2}))$

$$= \frac{9}{2} e$$

ii) $\int f(x) dx$

$$= \int 3x e^{-2x} dx$$

$$u = 3x \quad \frac{dv}{dx} = e^{-2x}$$

$$\frac{du}{dx} = 3 \quad \frac{dv}{dx} = -\frac{1}{2} e^{-2x}$$

$$\int f(x) dx = uv - \int v \frac{du}{dx} dx$$

$$= 3x \times -\frac{1}{2} e^{-2x} - \int -\frac{1}{2} e^{-2x} \times 3 dx$$

$$= -\frac{3}{2} x e^{-2x} + \int \frac{3}{2} e^{-2x} dx$$

$$= -\frac{3}{2} x e^{-2x} - \frac{3}{4} e^{-2x}$$

$$= \frac{3}{4} e^{-2x} (2x - 1)$$

$$\int_{-\frac{1}{2}}^0 f(x) dx = \left[\frac{3}{4} e^{-2x} (2x - 1) \right]_{-\frac{1}{2}}^0$$

$$= \left[\frac{3}{4} e^{-2(0)} (2(0) - 1) \right] - \left[\frac{3}{4} e^{-2(-\frac{1}{2})} (2(-\frac{1}{2}) - 1) \right]$$

$$= -\frac{3}{4} + \frac{3}{2} e$$

$$= \frac{3}{2} e - \frac{3}{4}$$

Item marks awarded: (i) = 2/3; (ii) = 3/5

Total mark awarded = 5 out of 8

Examiner comments

- (i) In both of these examples, the candidates have recognised the notation $f'(x)$ correctly as the derivative of $f(x)$ with respect to x , and they have then gone on to apply the product rule correctly.

Candidate 1 has found a correct answer, but they have not simplified their final answer to the expected final form. They have left their answer as $3e + 3e$, not as $6e$.

Candidate 2 has made an error before substituting for x , by thinking that $3x \times -2 = -3x$, but they have been awarded marks for the correct unsimplified expression for the derivative.

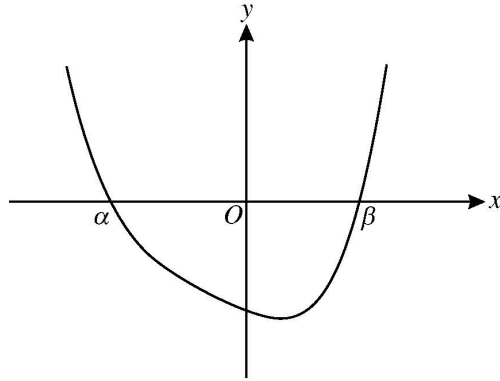
- (ii) This part of the question requires the use of integration by parts.

The factors and the signs complicated the task, but candidate 1 reached a correct unsimplified form of the answer. They proceeded to use the correct limits properly, but then made a sign error in simplifying their final answer when $0 - \frac{3e}{4}$ became $\frac{3e}{4}$.

Candidate 2 completed the first stage of the integration correctly, but then made a sign error in the course of simplifying their expression, obtaining $\frac{3}{2}xe^{-2x}$ when they should have had $-\frac{3}{2}xe^{-2x}$, so their answer is not correct at the second stage of integration. Although they earned the method mark for going on to use the correct limits appropriately, they were unable to obtain the correct final answer.

Question 6

6



The diagram shows the curve $y = x^4 + 2x^3 + 2x^2 - 4x - 16$, which crosses the x -axis at the points $(\alpha, 0)$ and $(\beta, 0)$ where $\alpha < \beta$. It is given that α is an integer.

- (i) Find the value of α . [2]
- (ii) Show that β satisfies the equation $x = \sqrt[3]{8 - 2x}$. [3]
- (iii) Use an iteration process based on the equation in part (ii) to find the value of β correct to 2 decimal places. Show the result of each iteration to 4 decimal places. [3]

Mark scheme

- 6 (i) Find y for $x = -2$ M1
Obtain 0 and conclude that $\alpha = -2$ A1 [2]
- (ii) Either Find cubic factor by division or inspection or equivalent M1
Obtain $x^3 + 2x - 8$ A1
Rearrange to confirm given equation $x = \sqrt[3]{8 - 2x}$ A1
- Or Derive cubic factor from given equation and form product with $(x - \alpha)$ M1
 $(x + 2)(x^3 + 2x - 8)$ A1
Obtain quartic $x^4 + 2x^3 + 2x^2 - 4x - 16 (= 0)$ A1
- Or Derive cubic factor from given equation and divide the quartic by the cubic M1
 $(x^4 + 2x^3 + 2x^2 - 4x - 16) \div (x^3 + 2x - 8)$ A1
Obtain correct quotient and zero remainder A1 [3]
- (iii) Use the given iterative formula correctly at least once M1
Obtain final answer 1.67 A1
Show sufficient iterations to at least 4 d.p. to justify answer 1.67 to 2 d.p. or show there is a change of sign in interval (1.665, 1.675) A1 [3]

Example candidate response – 1

6) (i) $y=0$, ~~$0 = x^4 + 2x^3 + 2x^2 - 4x - 16$~~
 ~~$x^4 + 2x^3 + 2x^2 - 4x = 16$~~
 ~~$x(x^3 + 2x^2 + 2x - 4) = 16$~~

$x^4 + 2x^3 + 2x^2 - 4x - 16 = 0$

by trial and error, $x = -1$, $y = -11$
 $x = -2$, $y = 0$

$\therefore \alpha = -2$

(ii) $x^4 + 2x^3 + 2x^2 - 4x - 16 = 0$
 $\frac{x^4}{2} + x^3 + x^2 - 2x - 8 = 0$

2

0

(ii)	$x = \sqrt[3]{8-2x}$
x_0	1
x_1	1.8171
x_2	1.6344
x_3	1.6788
x_4	1.6682
x_5	1.6707
x_6	1.6701
x_7	1.6703
x_8	1.6702

$\therefore x = 1.67$

3

3

Item marks awarded: (i) = 2/2; (ii) = 0/3; (iii) = 3/3

Total mark awarded = 5 out of 8

Example candidate response – 2

6(1)	$x^4 + 2x^3 + 2x^2 - 4x - 16 = 0$		
	$x^3(x+2) - 2x(x+2) = 16$		
	$\therefore (x^3 - 2x)(x+2) - 16 = 0$		
	$x^3 - 2x = 16$		
	$x + 2 = 16$		
	$(x+2) - 16 = 0$		
	$x = 14$		
	$\therefore x = 14$ *	X	M Q

Example candidate response – 2, continued

(ii) $(x^3 - 2x) - 16 = 0$
 $x = 16 + 2x$
 $x = \sqrt[3]{8 - 2x} = x$
 $8 - 2x = x^3$
 $16 - 2x = 2x^3$
 $x^4 + 2x^3 + 2x^2 - 4x - 16 = 0$
 $x^2(x^3 + 2x^2)$
 $x^3(x + 2x) + 2(x^2 - 2x - 16) = 0$
 $x^3(x + 2) +$
 (i) $(2x^3 - 4x - 16) + (x^3 + 2x^2) = 0$
 $x^2(2x + 2) + x(x^2 + 2) = 0$
 $x(2x^3 + 2x - 16) + x(x^3 + 2x^2) = 0$

(iii)
 $x = \sqrt[3]{8 - 2x}$
 $x_1 = 0$
 $x_2 = 2$
 $x_3 = 1.5874$
 $x_4 = 1.6899$
 $x_5 = 1.6655$
 $\beta = 1.67$

(2)

Item marks awarded: (i) = 0/2; (ii) = 0/3; (iii) = 2/3

Total mark awarded = 2 out of 8

Examiner comments

- (i) Candidate 1 has done what was expected, and used the factor theorem to test possible values for α . They have found that $y = 0$ when $x = -2$ and reached the correct conclusion.

Candidate 2 has looked for a factor of y , but a sign error in their working where $2x^2 - 4x$ became $-2x(x + 2)$ means that they do not reach their goal, leading them to draw an incorrect conclusion from false working.

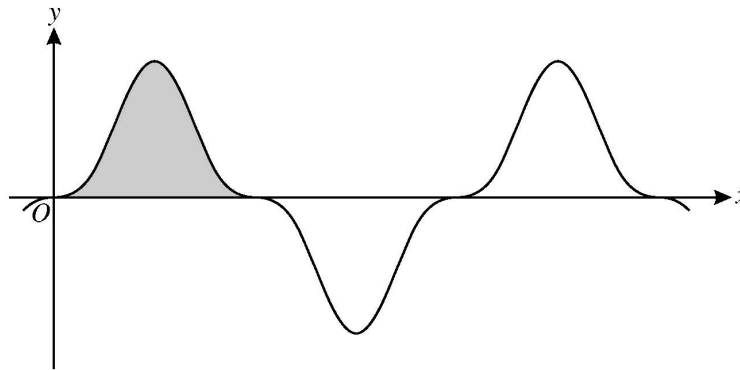
- (ii) Having found a value for α , the two possible approaches to part (ii) of the question were to divide y by $(x - \alpha)$ or to work back from the given answer. Candidate 1 has not recognised the potential to use either of these approaches.

Candidate 2 has crossed through their work, but they have clearly started from the given result, deduced correctly that this is the same as $8 - 2x = x^3$ and tried again to find a factor of y , but with no success.

- (iii) In both examples the candidates have gone on to use the given iterative formula correctly. Candidate 2 has reached the correct conclusion, but they have not given sufficient evidence to support this; they needed to have at least two consecutive values both supporting the same conclusion. Candidate 1 has made sure of their answer by continuing the process beyond the point (at x_5) where they could have reached the correct conclusion.

Question 7

7



The diagram shows part of the curve $y = \sin^3 2x \cos^3 2x$. The shaded region shown is bounded by the curve and the x -axis and its exact area is denoted by A .

- (i) Use the substitution $u = \sin 2x$ in a suitable integral to find the value of A . [6]
- (ii) Given that $\int_0^{k\pi} |\sin^3 2x \cos^3 2x| dx = 40A$, find the value of the constant k . [2]

Mark scheme

- 7 (i) State or imply $du = 2\cos 2x dx$ or equivalent B1
 Express integrand in terms of u and du M1
 Obtain $\int \frac{1}{2}u^3(1-u^2) du$ or equivalent A1
 Integration to obtain an integral of the form $k_1u^4 + k_2u^6, k_1, k_2 \neq 0$ M1
 Use limits 0 and 1 or (if reverting to x) 0 and $\frac{1}{4}\pi$ correctly DM1
 Obtain $\frac{1}{24}$, or equivalent A1 [6]
- (ii) Use 40 and upper limit from part (i) in appropriate calculation M1
 Obtain $k = 10$ with no errors seen A1 [2]

Example candidate response – 1

(7)(1) $y = \sin^2 2x \cos^3 2x$ $2 \sin 2a = 2 \sin a \cos a$
 $y = \frac{1}{2} \sin^3 4x$

$u = \sin 2x$
 $\frac{du}{dx} = 2 \cos 2x$ ✓

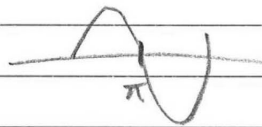
B1 $\frac{dx}{du} = \frac{1}{2 \cos 2x}$
~~dx~~ $\frac{du}{2 \cos 2x} = dx$

~~dx~~ \sin $\cos 4x = 1 - 2 \cos^2 2x$

A $y = \int (\sin 2x)^3 \cos^3 2x \left(\frac{du}{\cos 2x} \right)$ ✓
 $= \int u^3 \cos^2 2x \frac{du}{\cos 2x}$

M1 $= \int u^3 \cos 2x du$
 $= \int u^3 (1 - u^2) du$ $\cos^2 2x = (1 - \sin^2 2x)$

A0 $= \int u^3 - u^5 du$

when $y=0$, ~~sin~~ $\sin^3 2x \cos^3 2x = 0$. 
 $\frac{1}{2} \sin^3 4x = 0$
 $4x = \sin^{-1} 0$
 $x = \frac{\pi}{4}$

M0 $u = \sin 2 \left(\frac{\pi}{4} \right)$
 $u = 1$

M0 $A = \int_0^1 u^3 - u^5 du$
 $= \left[\frac{u^4}{4} - \frac{u^6}{6} \right]_0^1$

M0 $= \left(\frac{1}{4} - \frac{1}{6} \right) - 0$
 $= 0.05 \text{ unit}^2$ ✓

Example candidate response – 1, continued

7(ii) $\int_0^{k\pi} |\sin^3 2x \cos^3 2x| dx = 40 \text{ A}$

$= 40 \times 0.05 = 2.$

$2 = \int_0^{k\pi} |\sin^3 2x \cos^3 2x| dx$

$= \int_0^{k\pi} \frac{1}{2} (\sin 4x)^3 dx$

~~$\frac{1}{2} \left(\frac{1}{4} \sin 4x - \frac{1}{5} \sin^3 4x \right) = 0$~~

~~$\frac{1}{2} \left(\frac{1}{4} \sin 4x - \frac{1}{5} \sin^3 4x \right) = 2$~~

~~$\frac{1}{2} \left(\frac{1}{4} \sin 4x - \frac{1}{5} \sin^3 4x \right) = 40$~~

~~$\frac{1}{2} \left(\frac{1}{4} \sin 4x - \frac{1}{5} \sin^3 4x \right) = 40$~~

~~$\frac{1}{2} \left(\frac{1}{4} \sin 4x - \frac{1}{5} \sin^3 4x \right) = 40$~~

Ans: ~~when $V=1$~~

when $V=1$, $\sin 2x = 1$

$2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

~~Area = Rate~~

$k = 40 \times \frac{\pi}{4}$

$= 10\pi$

Ans: $k = 10$ //

2

(4)

Item marks awarded: (i) = 2/6; (ii) = 2/2

Total mark awarded = 4 out of 8

Example candidate response – 2

Question Numbers	
7.	i $y = \sin^3 2x \cos^3 2x$
B1	$u = \sin 2x$ $u^2 = \sin^2 2x = (1 - \cos^2 2x)$ $\frac{du}{dx} = 2 \cos 2x$
	$y = \int u \sin^3 2x \cos^3 2x dx$ $= \int \sin^3 2x \cdot \cos^3 2x \frac{du}{2 \cos 2x}$
	$= \int \sin^3 2x \cdot 2 \cos^2 2x du$
M1	$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^3 \cdot 2(1-u^2)^2 du$
AO1	$= \int_0^{\frac{\pi}{2}} 2u^3(1-u^2)(1-u^2) du$
	$= \left. \frac{2u^4}{4} - \frac{4u^6}{6} + \frac{2u^8}{8} \right _0^{\frac{\pi}{2}}$
M0	$= \frac{1}{2}u^4 - \frac{2}{3}u^6 + \frac{1}{4}u^8 \Big _0^{\frac{\pi}{2}}$
M0	$= (3.044 - 10.01 + 9.266) - 0 = 2.2956$
AO1	$\int_0^{k\pi} \sin^3 2x \cos^3 2x dx = 40A$
	$\int_0^{k\pi} \frac{1}{2} \sin^3 4x dx = 92.91824$
	$-\frac{1}{8} \cos^3 4x \Big _0^{k\pi} = 92.91824$
	$-\frac{1}{8} \cos^3 4k\pi + \frac{1}{8} = 92.91824$
	$\cos^3 4k\pi = -735 \dots -734.592$ $-\frac{1}{8} \cos^3 4k\pi = 91.699$ $\cos^3 4k\pi = -733.592$

Item marks awarded: (i) = 2/6; (ii) = 0/2

Total mark awarded = 2 out of 8

Examiner comments

- (i) This part of the question gives a clear indication of the method to be used. Candidate 2 has started correctly. They have a correct expression for $\frac{du}{dx}$ and have gone on to attempt to substitute for x in the integral. An algebraic slip where $\frac{\cos^3 2x}{2 \cos 2x}$ becomes $2 \cos^4 2x$ means that their substitution is incorrect and the subsequent method marks are not available because the terms are not of the required form.

Candidate 1 has also made a correct start to the substitution, but they have lost a factor of $\frac{1}{2}$ in the course of making the substitution. They have gone on to attempt the integration, but $\int u^5 du$ became $\frac{u^5}{5}$, so they have not been awarded further marks.

- (ii) This part of the question is asking the candidates to consider the symmetry of the function, but candidate 2, in common with many others, has attempted to find k by considering what happens when $k\pi$ is substituted as the upper limit of the integral. They have attempted the integral via an alternative route, using the double angle formula, but have made an error in claiming that $\int \frac{1}{2} \sin^3 4x dx = -\frac{1}{8} \cos^4 4x$.

This leads to an impossible equation in k and they stop.

In the first instance, candidate 1 also considers how to obtain $40A$ by changing the upper limit of the integral, but on reflection they then go back to consider the symmetry of the function and proceed to deduce the correct solution.

Question 8

8 Two lines have equations

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} p \\ 4 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix},$$

where p is a constant. It is given that the lines intersect.

- (i) Find the value of p and determine the coordinates of the point of intersection. [5]
- (ii) Find the equation of the plane containing the two lines, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. [5]

Mark scheme

- 8 (i) State or imply general point of either line has coordinates $(5 + s, 1 - s, -4 + 3s)$ or $(p + 2t, 4 + 5t, -2 - 4t)$ B1
 Solve simultaneous equations and find s and t M1
 Obtain $s = 2$ and $t = -1$ or equivalent in terms of p A1
 Substitute in third equation to find $p = 9$ A1
 State point of intersection is $(7, -1, 2)$ A1 [5]
- (ii) Either Use scalar product to obtain a relevant equation in a, b, c
 e.g. $a - b + 3c = 0$ or $2a + 5b - 4c = 0$ M1
 State two correct equations in a, b, c A1
 Solve simultaneous equations to obtain at least one ratio DM1
 Obtain $a : b : c = -11 : 10 : 7$ or equivalent A1
 Obtain equation $-11x + 10y + 7z = -73$ or equivalent with integer coefficients A1
- Or 1 Calculate vector product of $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$ M1
 Obtain two correct components of the product A1
 Obtain correct $\begin{pmatrix} -11 \\ 10 \\ 7 \end{pmatrix}$ or equivalent A1
 Substitute coordinates of a relevant point in $\mathbf{r} \cdot \mathbf{n} = d$ to find d DM1
 Obtain equation $-11x + 10y + 7z = -73$ or equivalent with integer coefficients A1
- Or 2 Using relevant vectors, form correctly a two-parameter equation for the plane M1
 Obtain $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$ or equivalent A1
 State three equations in x, y, z, λ, μ A1
 Eliminate λ and μ DM1
 Obtain $11x - 10y - 7z = 73$ or equivalent with integer coefficients A1 [5]

Example candidate response – 1

QUESTION 8

$$r_1 = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} p \\ 4 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} p \\ 4 \\ -2 \end{pmatrix} = 0$$

X

M 0

$$5p + 4 + 8 = 0$$

$$p = -\frac{12}{5}$$

$$r_1 = \begin{pmatrix} 5 + s \\ 1 - s \\ -4 + 3s \end{pmatrix}$$

$$r_2 = \begin{pmatrix} -\frac{12}{5} + 2t \\ 4 + 5t \\ -2 - 4t \end{pmatrix}$$

B 1

$$5 + s = -\frac{12}{5} + 2t \quad \text{--- ①}$$

$$1 - s = 4 + 5t \quad \text{--- ②}$$

$$-4 + 3s = -2 - 4t$$

$$\text{①} : s = -7\frac{2}{5} + 2t$$

M 1

$$\text{①} \rightarrow \text{②}$$

$$1 - 7\frac{2}{5} + 2t = 4 + 5t$$

$$-10\frac{2}{5} = 3t$$

$$t = -3\frac{7}{15}$$

$$s = -7\frac{2}{5} + 2\left(-3\frac{7}{15}\right)$$

$$s = -14\frac{1}{3}$$

A 0

$$\therefore \text{intersection pt} = \begin{pmatrix} 5 - 14\frac{1}{3} \\ 1 + 14\frac{1}{3} \\ -4 + 3(-14\frac{1}{3}) \end{pmatrix} = \begin{pmatrix} -9\frac{1}{3} \\ 15\frac{1}{3} \\ -47 \end{pmatrix}$$

A 0

Example candidate response – 1, continued

(ii) $\vec{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$

$$\vec{r} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 5 & -4 \end{vmatrix}$$

$$\vec{r} = \begin{pmatrix} 4 - 15 \\ 6 + 4 \\ 5 + 2 \end{pmatrix}$$

3 $\vec{r} = \begin{pmatrix} -11 \\ 10 \\ 7 \end{pmatrix}$ ✓

$$d = \begin{pmatrix} -11 \\ 10 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$$

$$d = -55 - 50 - 28$$

W $d = -33$

AO $\therefore -11x + 10y + 7z = -33$

6

Item marks awarded: (i) = 2/5; (ii) = 4/5

Total mark awarded = 6 out of 10

Example candidate response – 2

Question Numbers	
8. (i)	$r = \begin{pmatrix} 5 + s \\ 1 - s \\ -4 + 3s \end{pmatrix} \quad r = \begin{pmatrix} p + 2t \\ 4 + 5t \\ -2 - 4t \end{pmatrix}$
B1	$5 + s = p + 2t$ $1 - s = 4 + 5t$ $s = p + 2t - 5$ $1 - (p + 2t - 5) = 4 + 5t$ $1 - p - 2t + 5 = 4 + 5t$ $-p - 2t + 6 = 4 + 5t$ $-p + 6 - 4 = 5t + 2t$ $-p + 2 = 7t$ $t = \frac{-p + 2}{7}$
	$-4 + 3s = -2 - 4t$ $t = \frac{-2 - 2}{7} \quad t = -\frac{4}{7}$
M	$-4 + 3s = -2 - 4t$ $-4 + 3\left(p + 2\left(\frac{-p - 2}{7}\right)\right) = -2 - 4\left(\frac{-p - 2}{7}\right)$
A0	$-4 + 3\left(p - \frac{2p - 4}{7}\right) = -2 + \frac{4p + 8}{7}$ $-4 + 3p - \frac{6p - 12}{7} = -2 + \frac{4p + 8}{7}$ $-28 + 21p - 6p + 12 = -2 + 4p + 8$ $21p - 6p - 4p = -2 + 8 - 12 + 28$
A0	$11p = 22$ $p = 2$
	$r = \begin{pmatrix} 5 + s \\ 1 - s \\ -4 + 3s \end{pmatrix}$
A0	$r = \begin{pmatrix} 5 + \frac{29}{7} \\ 1 + \frac{29}{7} \\ -4 + 3\left(\frac{-29}{7}\right) \end{pmatrix}$
	<p>points of intersection</p> $r = \begin{pmatrix} 6/7 \\ 36/7 \\ -57/7 \end{pmatrix}$

Example candidate response – 2, continued

points of intersection = $\begin{pmatrix} 6 \\ 36 \\ -57 \end{pmatrix}$

8 (ii) $d_1 \cdot d_2 = n$

$$n = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 5 & -4 \end{pmatrix}$$

$$n = \begin{pmatrix} -1 & 3 \\ 5 & -4 \end{pmatrix} \bar{i} + \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} \bar{j} + \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix} \bar{k}$$

$$n = (4 - 15) \bar{i} + (6 + 4) \bar{j} + (5 + 2) \bar{k}$$

$$n = -11\bar{i} + 10\bar{j} + 7\bar{k}$$

M |
A |
A |

$$r \cdot n = AB \cdot n$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -11 \\ 10 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 36 \\ -57 \end{pmatrix} \cdot \begin{pmatrix} -11 \\ 10 \\ 7 \end{pmatrix}$$

not appropriate to use 7×1 M 0

$$-11x + 10y + 7z = -66 + 360 - 399$$

$$-11x + 10y + 7z = -105$$

A 0

(5)

Item marks awarded: (i) = 2/5; (ii) = 3/5

Total mark awarded = 5 out of 10

Examiner comments

- (i) Candidate 2 has used a correct method for answering this question, but in the sixth line of their working they have $6 - 4 = -2$, and so all of the answers that follow are incorrect.

Candidate 1 has also stated correct equations in s and t , but their response has gone wrong because they have started with a value of p found from an incorrect assumption (that the position vectors of the two given points must be perpendicular).

- (ii) The simplest way to find a vector perpendicular to both lines is to use the vector product, as these two candidates have done.

Having completed the vector product correctly, candidate 1 used one of the given points to attempt to find the value of d , but their arithmetic error $10 \times 1 = 50$ results in an incorrect final answer.

Candidate 2 has also tried to substitute the position vector of a point on the plane, but instead of using the point found in part (i) they have started by multiplying the coordinates by 7, and consequently their method is incorrect because the point they use does not lie on the plane.

Question 9

- 9 (i) Express $\frac{9 - 7x + 8x^2}{(3 - x)(1 + x^2)}$ in partial fractions. [5]
- (ii) Hence obtain the expansion of $\frac{9 - 7x + 8x^2}{(3 - x)(1 + x^2)}$ in ascending powers of x , up to and including the term in x^3 . [5]

Mark scheme

- 9 (i) State or imply form $\frac{A}{3 - x} + \frac{Bx + C}{1 + x^2}$ B1
 Use relevant method to determine a constant M1
 Obtain $A = 6$ A1
 Obtain $B = -2$ A1
 Obtain $C = 1$ A1 [5]
- (ii) Either Use correct method to obtain first two terms of expansion
 of $(3 - x)^{-1}$ or $\left(1 - \frac{1}{3}x\right)^{-1}$ or $(1 + x^2)^{-1}$ M1
 Obtain $\frac{A}{3}\left(1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3\right)$ A1
 Obtain $(Bx + C)(1 - x^2)$ A1
 Obtain sufficient terms of the product $(Bx + C)(1 - x^2)$, $B, C \neq 0$ and add the two expansions M1
 Obtain final answer $3 - \frac{4}{3}x - \frac{7}{9}x^2 + \frac{56}{27}x^3$ A1
- Or Use correct method to obtain first two terms of expansion
 of $(3 - x)^{-1}$ or $\left(1 - \frac{1}{3}x\right)^{-1}$ or $(1 + x^2)^{-1}$ M1
 Obtain $\frac{1}{3}\left(1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3\right)$ A1
 Obtain $(1 - x^2)$ A1
 Obtain sufficient terms of the product of the three factors M1
 Obtain final answer $3 - \frac{4}{3}x - \frac{7}{9}x^2 + \frac{56}{27}x^3$ A1 [5]

Example candidate response – 1

Question 9

i) $\frac{9-7x+8x^2}{(3-x)(1+x^2)} = \frac{A}{3-x} + \frac{Bx+C}{1+x^2}$ ✓

$$9-7x+8x^2 = A(1+x^2) + (Bx+C)(3-x)$$

B 1

Let $x=3$

$$60 = 10A$$

$$A = 6$$

M 1
A 1

Let $x=0$

$$9 = 6 + 3C$$

$$C = 1$$

A 1

Let $x=1$

$$4 = 12 + 2B + 4C$$

$$B = -3$$

A 0

∴ $\frac{9-7x+8x^2}{(3-x)(1+x^2)} = \frac{6}{3-x} + \frac{-3x+1}{1+x^2}$

ii) $\frac{4-7x+8x^2}{(3-x)(1+x^2)} = \frac{6}{3-x} + \frac{-3x+1}{1+x^2}$

$$\frac{6}{3-x} = \frac{6}{3(1-\frac{1}{3}x)}$$

$$= 2(1-\frac{1}{3}x)^{-1}$$

$$= 1 + -1(-\frac{1}{3}x) + \frac{-1(-2)}{2}(-\frac{1}{3}x)^2 + \frac{-1(-2)(-3)}{3!}(-\frac{1}{3}x)^3 + \dots$$

$$= 1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3$$

M 1
A 0

$$(1+x^2)^{-1} = 1 - 1(x^2) + \dots$$

$$= 1 - x^2$$

A 1

$$(-3x+1)(1-x^2) = -3x + 3x^3 + 1 - x^2$$

$$= 1 - 3x - x^2 + 3x^3$$

A 1

∴ $\frac{9-7x+8x^2}{(3-x)(1+x^2)} = \frac{6}{3-x} + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 - 3x - x^2 + 3x^3$

$$= \frac{2}{3} - \frac{8}{3}x - \frac{10}{9}x^2 + \frac{82}{27}x^3$$

M 1
A 0

(7)

Item marks awarded: (i) = 4/5; (ii) = 3/5

Total mark awarded = 7 out of 10

Example candidate response – 2

9.1)	$\frac{9-7x+8x^2}{(3-x)(1+x^2)} = \frac{A}{(3-x)} + \frac{B}{(1+x^2)}$	
	$9-7x+8x^2 = A(1+x^2) + B(3-x)$	
	$x=3$ $x=0$	BO
	$60 = 10A$ $9 = 6 + 3B$	
	$A=6$ $3 = 3B$	M1
		$B=1$
	$\frac{9-7x+8x^2}{(3-x)(1+x^2)} = \frac{6}{(3-x)} + \frac{1}{(1+x^2)}$	AO

Example candidate response – 2, continued

Question Numbers	9(ii)
	$\frac{6}{(3-x)} + \frac{1}{(1+x^2)}$
	$= 6(3-x)^{-1} + (1+x^2)^{-1}$
	$(3-x)^{-1}$
	$\Rightarrow \frac{1}{3}$
	$= \frac{1}{3}(1 - \frac{1}{3}x)^{-1}$
	$= \frac{1}{3} \left(1 + \frac{1}{3}x + \frac{(-1)(-2)}{2} \left(\frac{1}{9}x^2 \right) + \frac{(-1)(-2)(-3)}{6} \left(-\frac{1}{27}x^3 \right) \right)$
	$= \frac{1}{3} \left(1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 \right)$
	$= \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \frac{1}{81}x^3$
M	$6(3-x)^{-1} = 2 + \frac{2}{3}x + \frac{2}{9}x^2 + \frac{2}{27}x^3$ ✓
A1	$(1+x^2)^{-1}$
	$= 1 - x^2 + \frac{(-1)(-2)}{2}(x^4) + \frac{(-1)(-2)(-3)}{6}(x^6)$
	$= 1 - x^2 + x^4 - x^6$
A1	$6(3-x)^{-1} + (1+x^2)^{-1}$
	$= \left(2 + \frac{2}{3}x + \frac{2}{9}x^2 + \frac{2}{27}x^3 \right) + (1 - x^2)$
M0	$= 3 + \frac{2}{3}x - \frac{7}{9}x^2 + \frac{2}{27}x^3$
	(4)

Item marks awarded: (i) = 1/5; (ii) = 3/5

Total mark awarded = 4 out of 10

Examiner comments

- (i) Candidate 2 has started with an incorrect form for the partial fractions. They have demonstrated a correct method for attempting to find the value of a coefficient, but are unable to score any accuracy marks.

Candidate 1 uses the correct form for the partial fractions and everything goes well until the candidate makes an error in finding the final coefficient. They have found $10 = 12 + 2B + 4$ when it should be $10 = 12 + 2B + 2$.

- (ii) Candidate 2 goes on to use their incorrect coefficients correctly to obtain the expansions of both fractions. Due to the error in the coefficients, the work at the final stage is not equivalent (because there is no need for the multiplication of algebraic expressions), so the final method mark is not available.

Candidate 1 goes on to use their coefficients correctly, but in the course of expanding $6(3 - x)^{-1}$ the factor of 2 is lost and a correct unsimplified expansion is not seen. They have earned the final method mark because they proceeded to complete the work correctly for their expansions.

Question 10

10 (a) Without using a calculator, solve the equation $iw^2 = (2 - 2i)^2$. [3]

(b) (i) Sketch an Argand diagram showing the region R consisting of points representing the complex numbers z where

$$|z - 4 - 4i| \leq 2. \quad [2]$$

(ii) For the complex numbers represented by points in the region R , it is given that

$$p \leq |z| \leq q \quad \text{and} \quad \alpha \leq \arg z \leq \beta.$$

Find the values of p , q , α and β , giving your answers correct to 3 significant figures. [6]

Mark scheme

10 (a) Expand and simplify as far as $iw^2 = -8i$ or equivalent B1
 Obtain first answer $i\sqrt{8}$, or equivalent B1
 Obtain second answer $-i\sqrt{8}$, or equivalent and no others B1 [3]

(b) (i) Draw circle with centre in first quadrant M1
 Draw correct circle with interior shaded or indicated A1 [2]

(ii) Identify ends of diameter corresponding to line through origin and centre M1
 Obtain $p = 3.66$ and $q = 7.66$ A1
 Show tangents from origin to circle M1
 Evaluate $\sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$ M1
 Obtain $\alpha = \frac{1}{4}\pi - \sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$ or equivalent and hence 0.424 A1
 Obtain $\beta = \frac{1}{4}\pi + \sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$ or equivalent and hence 1.15 A1 [6]

Example candidate response – 1

10. (a)

$$i w^2 = (2 - 2i)^2$$

$$i w^2 = 4 - 8i + 4i^2$$

$$i w^2 = 4 - 8i - 4$$

$$i w^2 = -8i$$

$$w^2 = \frac{-8i}{i}$$

$$w^2 = 8$$

$$w = \sqrt{8}$$

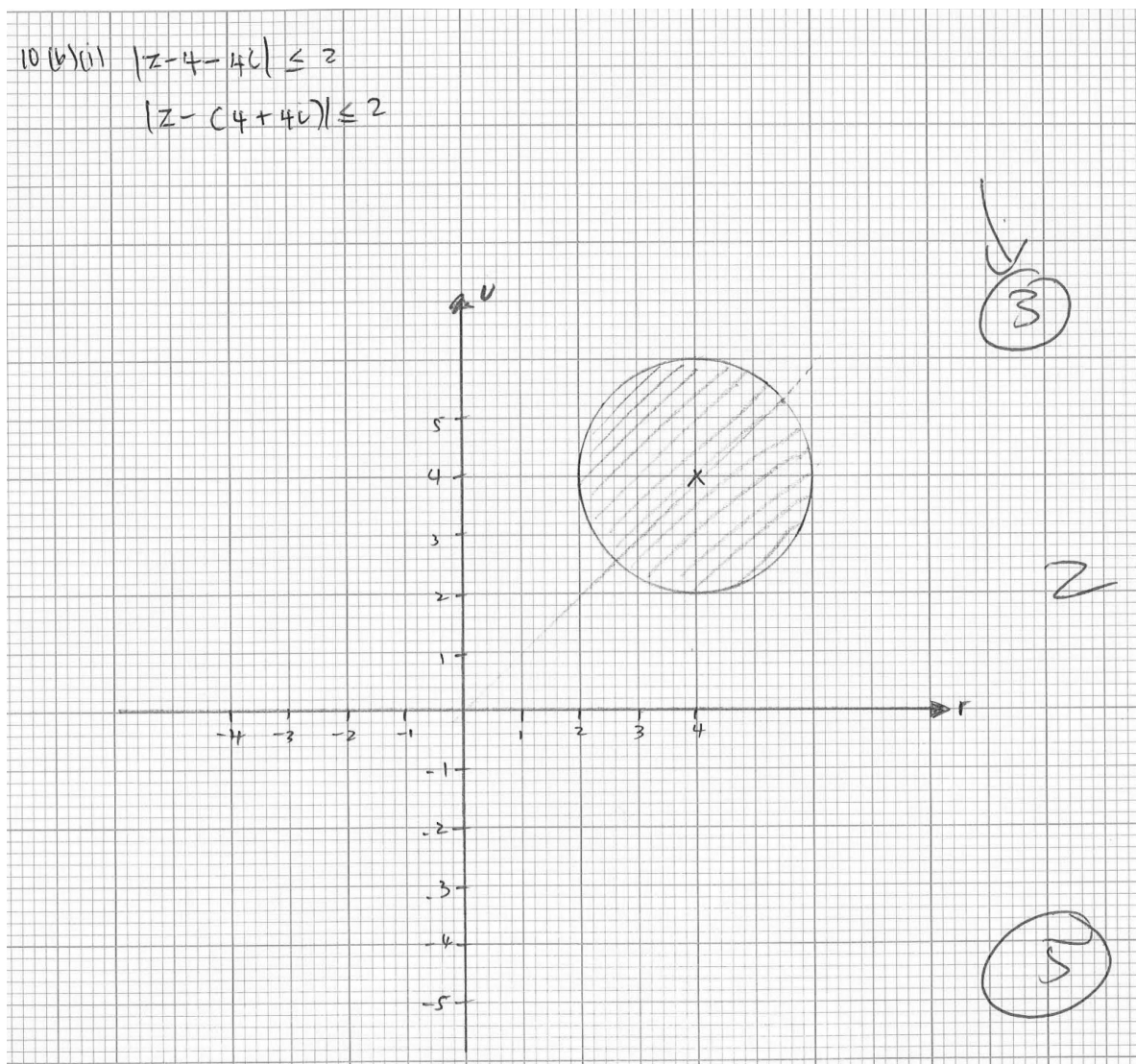
$$w = \sqrt{2 \times 4} \times \sqrt{-1}$$

$$w = 2\sqrt{2}i$$

$z = 4 - 4i$

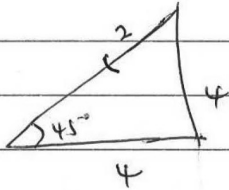
B 1

B 1



Example candidate response – 1, continued

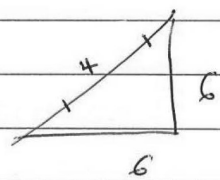
10(b)(ii) ~~in #~~



$$c = \sqrt{4^2 + 4^2}$$

$$p = \sqrt{32} - 2$$

$$p = 3.656854249$$

$$p = 3.657 //$$


$$c = \sqrt{6^2 + 6^2}$$

$$q = \sqrt{72} - 4$$

$$q = 4.485281374$$

$$= 4.49 //$$

M 1

A 0

$$\tan \theta = \frac{4}{4}$$

$$\theta = \frac{\pi}{4}$$

$$\text{avg } \tau = , \frac{\pi}{4}, \frac{3\pi}{4}$$

$$= 0.785398163, 2.35619449$$

$$= 0.785, 2.36$$

X

MO

$$\alpha = 0.785 //$$

$$\beta = 2.36 //$$

AO

Item marks awarded: (a) = 2/3; (b)(i) = 2/2; (b)(ii) = 1/6

Total mark awarded = 5 out of 11

Example candidate response – 2

10.	(a) $i w^2 = (2-2i)^2$ $i w^2 = 4 - 8i + 4i^2$ $= 4 - 8i - 4$ $i w^2 = -8i$ ✓	
B1	$-w^2 = -8$ $w = \sqrt{-8}$ $w^2 = 8$ $w = \sqrt{8}$ ✓	
BC		
	(b) (i) $ 2-4-4i \leq 2$ $ x+yi-4-4i \leq 2$ $ (x-4)+(y-4)i \leq 2$ $(x-4)^2 + (y-4)^2 \leq 2^2$ centre = (4,4) radius = 2	
2		
	(ii) $\arg z = \theta = \tan^{-1}\left(\frac{4}{4}\right)$ $= \tan^{-1} 1$ $= 0.785 \text{ rad}$ $\alpha = \tan^{-1}\left(\frac{2}{4}\right)$ $= 0.464 \text{ rad}$ $\beta = \tan^{-1}\left(\frac{4}{2}\right)$ $= 1.11 \text{ rad}$	$0.464 \leq \arg z \leq 1.11$
	$p \leq z \leq q$ $p = \sqrt{(2-0)^2 + (4-0)^2}$ $= \sqrt{20}$ $= 4.47$	$q = \sqrt{(4-0)^2 + (6-0)^2}$ $= \sqrt{52}$ $= 7.21$ ✗
	$\therefore 4.47 \leq z \leq 7.21$ $\therefore p = 4.47, q = 7.21, \alpha = 0.464, \beta = 1.11$	
(3)		

Item marks awarded: (a) = 1/3; (b)(i) = 2/2; (b)(ii) = 0/6

Total mark awarded = 3 out of 11

Examiner comments

- (a) Candidate 2 has reached $i\omega_2 = -8i$ correctly, but makes an error in dividing through by i . Because of this, their only solution for ω is incorrect.

Candidate 1 has a slip in their working, but there is sufficient working to support their conclusion that $\omega = \sqrt{-8}$. They have then proceeded to state the correct value for one of the two roots.

- (b) (i) Both candidates have drawn correct sketches to illustrate the region R . Candidate 2's sketch takes account of the two different scales. Candidate 1 used graph paper, which helps the candidate but was not a requirement.
- (ii) A correct diagram should have helped candidates to decide where to look to find the values of p , q , α and β .

Candidate 2 has done some work using modulus and argument. Some of the points used are shown on the diagram, but none of them fulfils the requirements of the question.

Candidate 1 has not shown the coordinates of any points on the diagram, but they have drawn the diameter of the circle passing through the origin. They have found the correct value for p , and their diagram confirms that this has been done correctly, but their working for q is not correct. They have found the angle between their diameter and the real axis correctly, but they do not show that they know which two angles they need to find to answer the question.

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