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FOREWORD

This booklet contains reports written by Examiners on the work of candidates in certain papers. **Its contents are primarily for the information of the subject teachers concerned**.

FURTHER MATHEMATICS

GCE Advanced Level

Paper 9231/01

Paper 1

General comments

The overall standard achieved by the candidates for this examination was very satisfactory and certainly an improvement on the performance of twelve months ago. In particular it is good to be able to record that the better candidates produced scripts which were impressive both in terms of understanding of concepts and working accuracy. From the less able, there was also some good work and even among them it was uncommon to see a script in which there had not been some attempt at every question. Thus for the most part substandard performance was primarily due to lack of technical focus and hence it is reasonable to conclude that most candidates had some knowledge of every syllabus topic.

There were almost no rubric infringements or misreads, and the working was usually, though not always, set out in a logical order. Not all scripts were easy to read in the mathematical sense, and this failure no doubt caused difficulties both for the Examiner and particularly for the candidate.

The syllabus topics with which candidates did best were polar coordinates, determination of asymptotes and curve sketching, transformation of polynomial equations, simple applications of de Moivre's theorem, and deriving the general solution of a second order linear differential equation. Topics which appeared to be the least well understood were convergence of infinite series, sums of integral powers of roots of polynomial equations in less straightforward cases, solving polynomial equations by trigonometric methods, mean values of higher derivatives over a finite interval, solution of a system of linear equations where the rank of the matrix of coefficients is less than the number of variables, some aspects of three dimensional metric vectors, evaluation of the coordinates of the centroid of a bounded plane region and matrix algebra.

Nevertheless candidates generally are to be congratulated on a positive attitude to this demanding examination and, in some cases, producing excellent work in a strictly time limited environment.

Comments on specific questions

Question 1

Over half the candidature produced a completely correct response to this question. A few candidates started off with an incorrect integral representation of the area of the loop, $\frac{1}{2}\int_0^{\pi/2} a^2 \sin 2\theta \, d\theta$, being the most popular variant. The key to the integration is to express $\sin^2 2\theta$ as $\frac{1-\cos 4\theta}{2}$ and most candidates effected this without error.

Answer: Area of loop = $\frac{\pi a^2}{8}$.

Question 2

About a half of all candidates answered this question without error. There were some notational confusions of N with n. A more serious error was the confusion of the inductive hypothesis with the result. The centre part of the argument is to show that $H_k \Rightarrow H_{k+1}$ for any integer positive integer k(A) and, as such, does not prove that H_k is true. The proof is completed by showing that H_1 is true (B). Of course, the stages (A) and (B) may be effected in either order. In the majority of responses the working at stage (A) was complete and accurate. Nevertheless, common errors here were the incorrect formation of term $k + 1$ and omission of essential detail in the subsequent working. At stage (B) which, evidently, some candidates thought was not worth bothering about, there was again lack of attention to detail.

Question 3

Most candidates appreciated that the evaluation of $S_N = \sum_{n=1}^{N} u_n$ could be effected by application of the difference method. Common incorrect answers were $v - (N + 1)v_{N+1}$ and $v_1 - (N + 2)v_{N+2}$.

(i) Only a minority wrote $S_N = 1 - (N + 1)^{\frac{1}{2}}$ before attempting to investigate the convergence (or otherwise) of $\sum_{n=1}^{\infty} u_n$. In this respect, an argument such as the following was expected. '$S_n = 1 - (N + 1)^{\frac{1}{2}} \to -\infty$ as $N \to +\infty$. Hence the infinite series $\sum_{n=1}^{\infty} u_n$ is not convergent.'

(ii) Likewise here, the corresponding argument should be as follows. '$S_N = 1 - (N+1)^{-\frac{1}{2}} \to 1 - 0 = 1$ as $N \to +\infty$'. Hence $\sum_{1}^{\infty} u_n$ is convergent and its sum to infinity is '1'. However, only a minority argued in this way. Thus the concept of convergence of an infinite series appears, overall, to have been poorly understood.

Question 4

This question was well answered by the majority of candidates.

Almost all responses showed the correct working to obtain the equations of the asymptotes.

The curve sketching, however, was less satisfactory. Common errors were failure to show both branches, failure to identify the points at which C crosses the coordinate axes, the locating of the maximum point of the lower branch on the y-axis and, especially, bad forms at infinity.

Answers: **(i)** $x = 3$ and $y = x + 3$; **(ii)** $(-2, 0)$, $(2, 0)$ and $(0, \frac{4}{3})$.

Question 5

Most candidates correctly made the substitution $x = \frac{y-1}{2}$ into the given equation in x (E1) and worked accurately to obtain the required equation in y (E2). Only a small minority used the incorrect substitution $x = 2y + 1$. Some even used E1 to evaluate symmetric functions of $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$ so as to obtain E2.

By no means everyone used E2 in a productive way. Thus it was not uncommon to see attempts to evaluate S_3 directly from E1 without any reference to E2. The algebra required for this strategy is demanding and those who were seriously involved in it must have found it very time consuming.

Likewise there were those who attempted to evaluate S_{-2} directly from $E1$, and this strategy involved yet more formidable algebra. One must remark, therefore, that it is not the aim of the A Level Further Mathematics examination to introduce complication for its own sake. Any candidate who is confronted with the prospect of protracted working should stop and look for the simple strategy that will lead to complete success.

In fact, from $E2$, one has almost immediately that $S_3 - S_1 - 3 = 0$ and $S_1 - S_{-1} - S_{-2} = 0$ from which, since $S_1 = 0$ and $S_{-1} = -1$, S_3 and S_{-2} can easily be obtained.

Answers: $S_3 = 3 : S_{-2} = 1$.

Question 6

In terms of the quality of candidate response, the two parts of this question contrasted sharply. Thus almost without exception, de Moivre's theorem was applied accurately in order to establish the displayed result for $\cos 6\theta$, whereas for the remainder of the question, working was often incomplete and inaccurate. Only about a quarter of all candidates produced a completely correct response.

There was a general awareness that with $x = \cos\theta$, the given x-equation is equivalent to an equation of the form $\cos 6\theta = \lambda$, where λ is a real number, but not everyone concluded that $\lambda = -\frac{1}{2}$ and even when they did, it was frequently the case that they were unable to make further progress. There were also those who wrote that the solutions took the form $\frac{k\pi}{9}$ for various integral k, without any mention of cosine. Among those who got as far as identifying the roots to be of the form $\cos \frac{k\pi}{9}$, there were some who gave k more than 6, even 12, values.

Answer: $\cos \frac{k\pi}{9}$, where $k = 1, 2, 4, 5, 7, 8$.

Question 7

(i) Almost all candidates used implicit differentiation in relation to the given x-y equation so as to obtain results relating x, y, y_1, y_2, where $y_n = \frac{d^n y}{dx^n}$. The first differentiation was almost always correct, but in the subsequent differentiation it was common to see '$\frac{\mathrm{d}}{\mathrm{d}x}\left(4y^3y_1\right) = 12y^2y_1 + 4y^3y_2$' rather than the correct $\frac{\mathrm{d}}{\mathrm{d}x}\left(4y^3y_1\right) = 12y^2y_1^2 + 4y^3y_2$. Beyond that, most candidates eliminated y_1 and then made intelligent use of $x^4 + y^4 = 1$ in an attempt establish the displayed result for y_2. Only about half of all candidates completed this part of the question.

(ii) In contrast, less than a quarter of all candidates went on to prove the second displayed result. Some wasted valuable examination time by attempting a third implicit differentiation so as to obtain y_3 in terms of x and y, only. The best candidates, however, wrote down a correct general formula for the mean value of y_3 over $a_1 \leqslant x \leqslant a_2$ and then saw at once that this is equal to $(a_2 - a_1)^{-1}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]_{a_1}^{a_2}$ and so went on to obtain the required result without apparent difficulty.

Question 8

This question marks the entry point into the more difficult part of the Paper so that it would be expected that completely correct responses here, and further on, would be less in evidence. Certainly this was the case with this question.

At the outset, almost all responses showed the accurate reduction of **A** to the echelon form and the correct determination of its rank. Beyond this initial stage, responses divided into two categories in the first of which the echelon form was used as a means of determining a basis β for the null space. Thus the better candidates, the minority, worked directly from equations $x - y - 2z + 3t = 0$, $y + 3z + 5t = 0$ and so established expeditiously a possible form for β. The less inspired, and much more error prone, method which was adopted by the majority, was to start with $\mathbf{Ax} = \mathbf{0}$ with the given form of **A** without reference to the echelon form previously obtained. This led to two independent linear equations in both of which there were no zero coefficients and many candidates were then unable to work accurately from this situation to a correct possible form of β.

For the final part, less than a quarter of the candidature worked from

$$\begin{pmatrix} p \\ q \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 5 \\ 0 \\ -1 \end{pmatrix},$$

though those who did so usually obtained the required result.

In contrast, the majority ignored the version of β obtained and so loaded themselves with a much more formidable task. They first evaluated $\mathbf{Ae}$ as $(2, -6, -4, -2)^{\mathrm{T}}$ and then went on to solve, in some way, the system $\mathbf{Ax} = (2, -6, -4, -2)^{\mathrm{T}}$ with $\mathbf{x}^{\mathrm{T}}$ preset to $(p, q, 1, 1)$. The success rate for this strategy was less than 50%.

Answers: Rank of **A** = 2,

A basis for the null space is $\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ 0 \\ -1 \end{pmatrix} \right\}$,

$p = -17$, $q = -18$.

Question 9

In the introductory part of this question, only about half of all candidates managed to carry out the required transformation of the given differential equation. Those who obtained $\dot{y} = 2x\dot{x}$ and $\ddot{y} = 2x\ddot{x} + 2\dot{x}^2$ had little difficulty in going on to the correct *t*-*y* differential equation. In contrast, those who started from $x = \sqrt{y}$ and then attempted to express $\dot{x}$ in terms of y and $\dot{y}$, and $\ddot{x}$ in terms of y, $\dot{y}$ and $\ddot{y}$, so as to effect the required transformation, usually ran into difficulties.

Almost all candidates obtained, or even wrote down without proof, the correct complementary function of the given *t*-*x* differential equation and had, at least, a correct overall strategy for obtaining the general solution. However, some candidates made elementary errors in the obtaining of the particular integral and these had a very negative effect both at this stage and in the remainder of the question.

Most candidates appeared to understand, in principle, how to apply the given initial conditions, so that the final stage of this question was, more than anything else, an accuracy test. In this respect, the least error prone strategy is to apply the initial conditions $y(0) = 4$, $\dot{y}(0) = -6$ to the general solution of the *t*-*y* equation so as to obtain the solution for $y(t)$. It is then a simple matter to obtain the required result from $x(t) = \sqrt{y(t)}$.

Those candidates who attempted to argue in this way, usually produced the correct result and certainly they did better than those who first obtained the general solution for $x(t)$ and then applied the initial conditions.

Answer: $x = [4e^{-3t} + 3\sin 2t]^{\frac{1}{2}}$.

Question 10

Most candidates used the scalar product to obtain a correct result for the acute angle between the two given planes, though a small minority obtained the obtuse angle and left it at that.

(i) The work here was hesitant even if, eventually, correct values of p and q did emerge from the working. Overall, there were many elementary errors.

(ii) Some candidates perceived that the direction of l can be obtained immediately from the result,

$(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} - \mathbf{k}) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,

but others used methods that indicated that they had no knowledge of the vector product. Nevertheless, over half of all candidates did obtain a correct vector equation of l.

In the final part of this question, most candidates understood that both Π_1 and Π_2 have equations of the form $x + 2y + 3z = d$, but comparatively few could obtain the correct value of d in either case. Thus the equation

$$\left|\frac{2-6+3-d}{\sqrt{14}}\right| = \sqrt{14},$$

or equivalent, seldom appeared.

Answers: Acute angle between given planes is 61.9°; **(i)** $p = 2$, $q = -3$; **(ii)** $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$; Equations of planes are $x + 2y + 3z = 13$, $x + 2y + 3z = -15$.

Question 11 EITHER

Responses, generally, showed understanding of the ideas involved, but lack of technical expertise, more prevalent here than anywhere else in this Paper, undermined most attempts.

Most responses showed sound working to establish the displayed reduction formula. Nevertheless, it must be remarked that the formation of mathematical symbols was frequently very poor and that in some scripts it was even difficult to distinguish between α and n. Almost all responses showed use of the integration by parts rule, and very few adopted the easier strategy which begins with $D(x^n e^{-\alpha x}) = nx^{n-1}e^{-\alpha x} - \alpha x^n e^{-\alpha x}$, and from which the required result follows almost immediately.

The most direct strategy for the obtaining of $\bar{x}$ is to set $\alpha = 1$ in the reduction formula and then to write $\bar{x} = \frac{I_2}{I_1} = 2 - \frac{e^{-1}}{I_1}$. Since $I_1 = I_0 - e^{-1} = \ldots = 1 - 2e^{-1}$, then the result is immediate. However, almost no candidates availed themselves of this simple strategy. Instead, many responses showed use of the reduction formula, again with $\alpha = 1$, to obtain I_1 and I_2 in terms of e and then went on to determine $\bar{x}$. There were also those who made no use of the reduction formula at all and so became involved in much protracted working. Responses usually identified the correct general formula for $\bar{y}$, but attempts to evaluate this coordinate in terms of e were frequently doomed at the outset by failure to understand that for I_2 the working value of α is now 2. Even where this was understood, working was often undermined by elementary errors, so that very few candidates obtained a correct result for both coordinates.

Answer: $\bar{x} = \dfrac{2e-5}{e-2}$; $\bar{y} = \dfrac{e^2-5}{8e(e-2)}$.

Question 11 OR

This question proved to be more popular and better worked than its alternative.

There were relatively few complete and correct proofs of the first result of the question. The majority of responses showed lack of understanding of the dimensions of the quantities and contained at least one meaningless equation such as '$\mathbf{AeBe} = \lambda\mathbf{e}\mu\mathbf{e}$'.

In contrast, the determination of the eigenvalues of **C** was effected accurately by almost all candidates who attempted this question and very often a correct set of eigenvectors to match eventually appeared.

Few candidates made significant progress with the final part. Some responses showed complicated attempts to obtain the eigenvalues of **D**. Few indicated any understanding of the obvious fact that they can be written down at sight. Some candidates thought that it was sufficient to show that **C** and **D** have a common eigenvalue which, in fact, they do, namely, –2. What was required was a listing of all the eigenvectors of **D** which, if previous working had been correct, would show immediately that an eigenvector common to both **C** and **D** is $(1 \ \ -2 \ \ 1)^T$, and hence that 2 x –4 = –8 is an eigenvalue of **CD**.

Answers: Eigenvalues of **C** are –2, 2, 4, with corresponding eigenvectors $\begin{pmatrix}17\\-6\\-7\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$.

An eigenvalue of **CD** is –8.

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Paper 2

General comments

The performance of candidates was very varied, with some producing excellent work and others exhibiting little or no grasp of the Syllabus. Most of those who produced reasonable work attempted all the questions, suggesting that the time pressure was not too great. Indeed a few candidates attempted both the alternatives in **Question 11**, though only the better of their two answers could be given any credit.

As is often the case in such examinations, some candidates failed to adequately explain their working, which can needlessly lose credit if their final answer is incorrect since no marks can be awarded for their method of solution. Examples of this are the first three parts of **Question 5**, where it is essential to explain how the moments of inertia are found, and the expected frequencies in **Question 7**, though other questions exhibited similar behaviour.

Comments on specific questions

Question 1

(i) This rarely presented any difficulty, and most candidates differentiated twice in order to produce the given equation.

(ii) Although simple harmonic motion was usually defined adequately, whether in words or by means of the relevant differential equation, remarkably few candidates showed that *P* is describing this motion. Some stated that the second derivative is proportional to *x*, which is clearly untrue, or ignored the requirement and instead simply found the *x*-coordinate of *A*. *P* is best shown to be describing SHM by putting $y = x - 12$ and noting that $\ddot{y} = -9y$. While the period was often found correctly, very few candidates could deduce the amplitude, with 17 being a popular though incorrect answer. The extreme point can be found if necessary by setting the velocity to zero.

Answers: **(ii)** 12; $\frac{2\pi}{3}$ s; 13 m.

Question 2

In principle the solution is straightforward, with the conservation of momentum and Newton's law of restitution being applied to the two collisions between the spheres, and only the latter equation to the collision of *B* with the barrier. While these principles were usually understood, many candidates introduced errors, often due to confusion over signs. It would have helped in some cases if they had defined the positive direction of the spheres' velocities explicitly, whether in words or by means of a sketch.

Answers: 0, $\frac{1}{2}U$ towards the barrier.

Question 3

The given equation is obtained by equating the gain in kinetic energy to the loss of potential energy, considering the two rings together. Most candidates appreciated that the two terms of interest in the second given equation represent the components of tension and weight respectively, though fewer identified these as the components in the transverse direction. The corresponding components – $T\cos\beta$ and $mg\sin(\beta+\theta)$ for Q were frequently mis-stated, though even those who did so usually appreciated that the final result is obtained by equating the expressions for the transverse acceleration components of P and Q.

Question 4

Neither part of this question was answered well on the whole. The required inequality in the first part follows from finding the friction F and normal reaction R at B, or even better their ratio by taking moments about C, and then using $F \leqslant \mu R$. A common fault was to take the frictional force instead as equal to μR, and then fail to justify the introduction of an inequality. The second part requires three independent moment or resolution equations in order to eliminate the other forces and derive P in terms of W, but many candidates either found too few equations, or introduced errors into the equations or the elimination process.

Answer: $\dfrac{4W}{81}$.

Question 5

As indicated in the general comments above, some candidates introduced various terms which combined to produce the required moments of inertia, but the lack of explanation made their working insufficiently convincing. Since there are different ways of producing the results in parts **(ii)** and **(iii)**, it is even more important to explain how the answers are arrived at.

(i) The List of Formulae gives the moment of inertia of a thin rod about its centre, but it is not valid to simply multiply this by 4 and wrongly take the mass of each rod to be M, as many candidates appeared to do. Instead the parallel axes theorem should be applied for the four rods, and the mass of each replaced by $\dfrac{M}{4}$. Some candidates wrongly used the formula for a rectangular lamina.

(ii) The required result is most easily obtained by applying the parallel axes theorem to the previously found moment of inertia.

(iii) One method of solution is to apply the perpendicular axes theorem at the centre of the body in order to find the moment of inertia parallel to BD, and then the parallel axes theorem to find the required moment of inertia through A, but it can also be found from the result in **(ii)**.

The angular speed follows from conservation of energy, as many candidates appreciated, but not all realised that the centre of the lamina falls a vertical distance of $a\sqrt{2}$.

Answer: $\sqrt{\dfrac{3g\sqrt{2}}{4a}}$.

Question 6

The great majority of candidates calculated the product moment correlation coefficient correctly, but many fewer noted that the low correlation makes the use of the regression line inadvisable. Instead some based their conclusion on the area of the seventh Asian country lying outside the range of the data, even though there was no indication in the question of this being the case.

Answer: -0.14.

Question 7

This question was usually answered well, resulting in the calculated value of χ^2 of about 4.36 (dependent on the number of significant figures retained in the working) being compared with the tabular value 7.815, and hence the conclusion that the Poisson distribution fits the data. The two most common errors were to calculate the expected frequency for exactly five cyclists, rather than five or more, and to fail to combine the last two cells.

Question 8

By contrast to the previous question, many candidates used entirely the wrong approach, failing to estimate a common population variance or to use the appropriate formula for the confidence interval with the tabular *t*-value 2.306. Even those who employed a broadly correct method often overlooked the fact that the given estimates of the population variances are said to be unbiased rather than biased. The assumptions of normal populations and common variance were rarely both stated.

Answer: 0.22 ± 0.155 .

Question 9

The initial probability, E(N), Var(N) and E(N^2) presented few problems. However errors were often introduced when finding the probability in part **(ii)**, such as calculating instead $P(N \leqslant a) - P(N = 1)$. The final part can be answered by, for example, listing all the possible outcomes with two consecutive wins and then summing their probabilities, but completely correct answers were rare.

Answers: $(1 - p)^{n-1}p$; **(i)** $\frac{1}{p}$, $\frac{1-p}{p^2}$, $\frac{2-p}{p^2}$; **(ii)** $1 - (1 - p)^a$.

Question 10

The appropriate test here is the paired sample one, but many candidates did not use it and therefore gained little credit. The calculated *t*-value of 1.93 is compared with 1.833 to conclude that the mean weight of the population is reduced at the 5% level, and then with 2.262 to conclude that there is no reduction at the 2.5% level of significance.

Question 11 EITHER

The application of Hooke's Law gives the initial extension immediately, and SHM is shown by noting that the net force responsible for the particle's acceleration comes from the tension and the weight. Some candidates wrongly attempted to include the impulse when finding this net force. The period and amplitude follow from the standard formulae, but care is needed in finding the elapsed time, since it is not simply three-quarters of the period as some candidates supposed. Instead it is necessary to express the displacement in terms of either a sine or cosine in order to find the time between the point at which the string becomes slack and the equilibrium point, or some equivalent approach.

Answers: 0.64; 1.59 s; 1.26 m; 0.929 s.

Question 11 OR

In general, candidates seemed to have more success in this question than in the alternative Mechanics one. F(x) is found by integrating f(x) with the appropriate limits, while E(X) is the integral of xf(x) over the semi-infinite range of x, and the median is found by equating F(x) to $\frac{1}{2}$. $P(Y < y)$ may be written down by replacing x by $\sqrt{y}$ in F(x), and differentiating the resulting expression gives the probability density function of Y. E(Y) is Var(X) + $E^2(X)$, and may be calculated by either noting that Var(X) is $\frac{1}{2}$ for such an exponential distribution, or by integrating x^2f(x). The final relationship between the two medians may be demonstrated by, for example, solving $P(Y < y) = \frac{1}{2}$ in order to find the median of Y.

Answers: **(ii)** $\frac{1}{2}\ln 2$; **(iii)** $1 - e^{-2\sqrt{y}}$, $\frac{e^{-2\sqrt{y}}}{\sqrt{y}}$.