UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1 **May/June 2007**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

0222815537

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

UNIVERSITY *of* CAMBRIDGE
International Examinations

1 Verify that

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{2n+1}{(n^2+1)(n^2+2n+2)}. \qquad [1]$$

Use the method of differences to show that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{2n+1}{(n^2+1)(n^2+2n+2)} < \frac{1}{2}. \qquad [3]$$

Write down the value of

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+1)(n^2+2n+2)}. \qquad [1]$$

2 The curve C is defined parametrically by

$$x = t - \ln t, \quad y = 4t^{\frac{1}{2}},$$

where $t > 0$. The arc of C joining the point where $t = 1$ to the point where $t = 4$ is rotated through one complete revolution about the x-axis. Show that the area of the surface generated is $\frac{160}{3}\pi$. [6]

3 Find the general solution of the differential equation

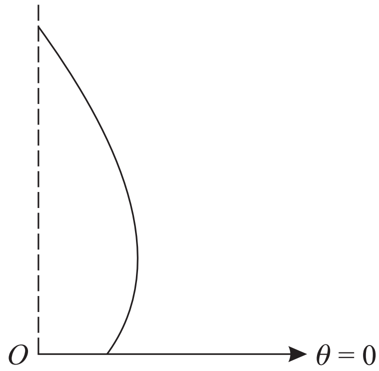
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 29y = 58x + 37. \qquad [6]$$

4 Given that the variables x and y are related by

$$y = x + e^{-xy},$$

find the value of $\frac{d^2y}{dx^2}$ when $x = 0$. [6]

5



The diagram shows the curve C with polar equation $r = e^{\theta}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Find the maximum distance of a point of C from the line $\theta = \frac{1}{2}\pi$, giving the answer in exact form. [4]

Find the area of the finite region bounded by C and the lines $\theta = 0$ and $\theta = \frac{1}{2}\pi$, giving the answer in exact form. [3]

6 The matrix $\mathbf{A}$, given by

$$\mathbf{A} = \begin{pmatrix} 7 & -4 & 6 \\ 2 & 2 & 2 \\ -3 & 4 & -2 \end{pmatrix},$$

has eigenvalues 1, 2, 4. Find a set of corresponding eigenvectors. [4]

Hence find the eigenvalues of $\mathbf{B}$, where

$$\mathbf{B} = \begin{pmatrix} 10 & -4 & 6 \\ 2 & 5 & 2 \\ -3 & 4 & 1 \end{pmatrix},$$

and state a set of corresponding eigenvectors. [3]

7 The equation

$$x^3 + 3x - 1 = 0$$

has roots α, β, γ. Use the substitution $y = x^3$ to show that the equation whose roots are $\alpha^3, \beta^3, \gamma^3$ is

$$y^3 - 3y^2 + 30y - 1 = 0.$$ [2]

Find the value of $\alpha^9 + \beta^9 + \gamma^9$. [5]

8 The sequence $x_1, x_2, x_3, \ldots$ is such that $x_1 = 1$ and

$$x_{n+1} = \frac{1 + 4x_n}{5 + 2x_n}.$$

Prove by induction that $x_n > \frac{1}{2}$ for all $n \geqslant 1$. [5]

Prove also that $x_n > x_{n+1}$ for all $n \geqslant 1$. [3]

9 Let

$$I_n = \int_0^1 \frac{1}{(4-x^2)^n}\,\mathrm{d}x,$$

for $n = 1, 2, 3, \ldots$. Verify that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{(4-x^2)^n}\right] = \frac{1-2n}{(4-x^2)^n} + \frac{8n}{(4-x^2)^{n+1}}$$

and hence prove that

$$8nI_{n+1} = (2n-1)I_n + \frac{1}{3^n}.$$ [5]

Find the y-coordinate of the centroid of the region bounded by the axes, the line $x = 1$ and the curve

$$y = \frac{1}{4-x^2},$$

giving your answer correct to 3 decimal places. [5]

10 The line l_1 passes through the points P and Q whose position vectors are $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $-2\mathbf{i} + 5\mathbf{j} + 13\mathbf{k}$ respectively. The line l_2 passes through the point S whose position vector is $\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ and is parallel to the vector $\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. The point whose position vector is $-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ is on the line l_3, the common perpendicular to l_1 and l_2.

(i) Find, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, an equation for l_3. [3]

(ii) Find the perpendicular distance from S to l_3. [4]

(iii) Find the perpendicular distance from S to the plane which contains l_3 and passes through P. [4]

11 **(a)** Use de Moivre's theorem to show that $\sin 8\theta$ can be expressed in the form

$$\sin\theta\cos\theta(a\sin^6\theta + b\sin^4\theta + c\sin^2\theta + d),$$

where the value of the constant a is to be found and b, c, d are constants whose values need not be found. [6]

(b) Use de Moivre's theorem to show that

$$\sum_{n=1}^{N} \frac{\sin n\theta}{2^n} = \frac{2^{N+1}\sin\theta + \sin N\theta - 2\sin(N+1)\theta}{2^N(5 - 4\cos\theta)}.$$ [7]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \lambda x + \frac{x}{x+2},$$

where λ is a non-zero constant.

(i) Find the asymptotes of C. [3]

(ii) Show that if $\lambda > 0$ then $\frac{dy}{dx} > 0$ at all points of C. [2]

(iii) Show that, for $\lambda < -\frac{1}{2}$, C has two distinct stationary points, both to the left of the y-axis. [3]

(iv) In separate diagrams draw sketches of C for each of the cases $\lambda > 0$ and $\lambda < -\frac{1}{2}$. [6]

OR

The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & 2 & 4 \\ 2 & -4 & 5 & 9 \\ 3 & -6 & 8 & 14 \\ 5 & -10 & 12 & 22 \end{pmatrix}.$$

(i) Find the rank of $\mathbf{M}$. [3]

(ii) Obtain a basis for the null space, K, of T. [3]

(iii) Evaluate

$$\mathbf{M}\begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix},$$

and hence show that any solution of

$$\mathbf{Mx} = \begin{pmatrix} 5 \\ 11 \\ 17 \\ 27 \end{pmatrix} \qquad (*)$$

has the form

$$\begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_2,$$

where λ and μ are constants and $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for K. [3]

(iv) Find the solution $\mathbf{x}_1$ of $(*)$ such that the first component of $\mathbf{x}_1$ is A, and the sum of all the components of $\mathbf{x}_1$ is B. [5]

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