UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2009 question paper

for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/01 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- CIE will not enter into discussions or correspondence in connection with these mark schemes.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

1 $x = y^{1/3} \Rightarrow y^4 = (1+y)^3$ M1A1

$\Rightarrow y^4 - y^3 - 3y^2 - 3y - 1 = 0$ A1

$\sum \alpha^6 = 1 - 2 \times (-3) = 7$ M1A1

or $y = x^2 \Rightarrow y^4 - y^3 - 2y^2 + 1 = 0$ has roots $\alpha^2,\ \beta^2,\ \gamma^2$ M1

$\sum \alpha^6 = \left(\sum \alpha^2\right)\left(\sum \alpha^4\right) - \left(\sum \alpha^2\right)\left(\sum \alpha^2\beta^2\right) + 3\left(\sum \alpha^2\beta^2\gamma^2\right)$ M1A2

$= 5 - (-2) + 0 = 7$ A1

or For last 2 marks

$z = y^2 \Rightarrow z^4 - 7z^3 + z^2 - 3z + 1 = 0$ M1

$\sum \alpha^6 = -\dfrac{(-7)}{1} = 7$ A1

or Use of $S_{N+4} = S_N + S_{N+3}$ M1

$S_{-1} = \dfrac{0}{1} = 0$ $S_2 = 1^2 - 2 \times 0 = 1$ (both) M1A1

$S_3 = 0 + 1 = 1$
$S_4 = 1 + 4 = 5$ (both) A1

$S_5 = 5 + 1 = 6$
$S_6 = 6 + 1 = 7$ (both) A1

2 Verifies displayed result M1A1

(i) $S_N = 1/15 - 1 / (N + 3)(2N + 5)$ M1A1A1

(ii) $S_\infty = 1 / 15$ A1 ft

Note: Must see working for preliminary result

Either $(2n^2 + 11n + 15) - (2n^2 + 7n + 6)$ (oe) } in numerator

Or $(n + 3)(2n + 5) - (n + 2)(2n + 3)$

A1 A1

3	$\bar{y} = \dfrac{\frac{1}{2}\int y^2 dx}{\int y dx}$	M1
	$\int_0^a y dx = \lambda a^3 / 3$	B1
	$\int_0^a y^2 dx = \lambda^2 a^5 / 5$	M1A1
	$\bar{y} = ... = 3\lambda a^2 / 10$	A1
	$\bar{y} = a \Rightarrow \lambda = \dfrac{10}{3a}$ (AG)	A1
	Finding $\bar{x}$ is not a misread and scores M0	
4	$dy / dx = x^2$ and range of x is $[0,1] \Rightarrow s = \int_0^1 (1 + x^4)^{1/2} dx$ (AG)	M1A1
	$S = 2\pi \int_0^1 (x^3 / 3 + 1)(1 + x^4)^{1/2} dx$ (AEF) (completely correct)	M1
	$S = 2\pi s + (2\pi / 3)\int_0^1 x^3 (1 + x^4)^{1/2} dx$	A1
	$\int x^3 (1 + x^4)^{1/2} dx = (1/6)(1 + x^4)^{3/2}$	B1
	$S = (\pi / 9)(18s + 2\sqrt{2} - 1)$ (AG) cwo	A1

5 Sketch with correct shape, location and orientation B1

Shows tangency to the initial line at the pole B1

Ignore extra in diagram

Draws, in the same diagram, a straight line passing through the origin and with positive gradient (distinct half-line and not a construction line) B1

$(1/2)\int_0^{\pi/2}\theta^2 d\theta = \pi^3/48$ M1A1

$(1/2)\int_0^{\alpha}\theta^2 d\theta = \alpha^3/6$ A1

$\alpha^3/6 = \pi^3/96 \Rightarrow \alpha = \pi.2^{-4/3}$ (aef) A1

or for previous 4 marks:

$(1/2)\int_0^{\alpha}\theta^2 d\theta = (1/2)\int_{\alpha}^{\pi/2}\theta^2 d\theta$ M1

$\alpha^3/6 = (1/6)[(\pi/2)^3 - \alpha^3]$ A1A1

$\Rightarrow \alpha^3 = \pi^3/16 \Rightarrow \alpha = \pi(16)^{-1/3}$, or equivalent A1

6 $(1 + y_1)(x^2 + y^2) + (x + y)(2x + 2yy_1) = 0$ B1B1

Puts $x = 0$, $y = 1$ to obtain $y_1 = -1/3$ B1

$(x^2 + y^2)y_2 + 2(1 + y_1)(2x + 2yy_1) + (x + y)(2 + 2y_1^2 + 2yy_2) = 0$ B1B1B1

Puts $x = 0$, $y = 1$, $y_1 = -1/3$ to obtain $y_2 = -4/9$ (cwo) B1

or

$x^3 + xy^2 + x^2y + y^3 = 1$ (expanding)

Differentiating

$\underbrace{3x^2 + y^2 + 2xyy'}_{\text{B1}} + \underbrace{2xy + x^2y' + 3y^2y'}_{\text{B1}} = 0$

Sub (0,1) $1 + 3y' = 0 \Rightarrow y' = -\frac{1}{3}$ B1

Differentiating again

$\underbrace{6x + 2yy' + 2yy' + 2x(yy'' + (y')^2)}_{\text{B1}} + \underbrace{2y + 2xy' + x^2y'' + 2xy'}_{\text{B1}} + \underbrace{6y(y')^2 + 3y^2y''}_{\text{B1}} = 0$

Sub (0,1) $\frac{4}{3} + 3y'' = 0 \Rightarrow y' = -\frac{4}{9}$ B1

For those who write $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\left(3x^2 + 2xy + y^2\right)}{\left(x^2 + 2xy + 3y^2\right)}$

4th, 5th and 6th marks are awarded as follows:

$$\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}(\text{B1}) - \frac{u\mathrm{d}v}{\mathrm{d}x}(\text{B1})}{v^2(\text{B1})}$$

7 $I_n = \left[-t^n e^{-t}\right]_0^1 + n\int_0^1 t^{n-1}e^{-t}\,dt$ M1A1

$I_n = nI_{n-1} - e^{-1}$ (AG) A1

$H_k : I_k < k!$ for some k B1

$H_k :\Rightarrow I_{k+1} < (k+1)k! - e^{-1} < (k+1)!$ M1A1

$I_1 = 1 - 2e^{-1}$ B1

(Do **not** allow I_0 unless I_1 is deduced from it)

Completes induction argument A1

This requires at least:

'The assumption is true and it is true for all positive integers' for the final mark, which is only awarded if all previous marks have been gained in the proof part.

8 Roots of AQE are $-1/2 \pm 4i$ (allow 1 sign/coefficient error) M1

$CF = e^{-x/2}(A\sin 4x + B\cos 4x)$ A1

PI of the form $ax^2 + bx + c$ M1

$\text{PI} = x^2 + 1$ M1A1

General solution: $y = e^{-x/2}(A\sin 4x + B\cos 4x) + x^2 + 1$ A1

$CF/x^2 \to 0\ \forall A$ and $B \Rightarrow CF/x^2 \to 0$ M1

$\Rightarrow y/x^2 \to 1$ as $x \to \infty$ (AG, CWO) A1

Allow $e^{-kx} \to 0$ as $x \to \infty$ provided $K > 0$ for M1

[Use of D operator for PI M1

Obtaining PI M1A1]

9 Exhibits $\mathbf{A} - \lambda\mathbf{I}$ in a numerical form, where $\lambda = 1, 5$ or 7 M1

Uses this form in some way to obtain1 eigenvector M1

Eigenvectors: any non-zero scaling of $\begin{pmatrix} 17 \\ -6 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ A1A1A1

$\mathbf{P} = \begin{pmatrix} 17 & 1 & 1 \\ -6 & -2 & 0 \\ -7 & 1 & 1 \end{pmatrix}$ Do not allow $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ B1 ft

$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 7^n \end{pmatrix}$ M1A1

$k^n\mathbf{A}^n = \mathbf{P}\begin{pmatrix} k^n & 0 & 0 \\ 0 & k^n5^n & 0 \\ 0 & 0 & k^n7^n \end{pmatrix}\mathbf{P}^{-1}$ M1A1

$-1/7 < k < 1/7$ A1

For first 2 marks

Forms ⩾ 1 vector product M1

Evaluates ⩾ 1 vector product M1

10	One asymptote is $x = -\lambda$	B1
	$y = x - \lambda + \lambda^2/(x + \lambda)$	M1
	Other asymptote is $y = x - \lambda$	A1
	or for previous 2 marks:	
	$mx + c = x^2/(x + \lambda) \Rightarrow (m - 1)x^2 + (c + \lambda m)x + \lambda c = 0$	
	$m - 1 = 0$, $c + \lambda m = 0$ (both)	M1
	$m = 1$, $c = -\lambda$ (both)	A1
	Differentiates and $y' = 0$	M1
	Identifies turning point at $(-2\lambda, -4\lambda)$ and $(0, 0)$	A1A1
	(Award D1 if correct turning points stated with no working)	
	First graph:	
	Axes and both asymptotes	B1
	Branches	B1B1
	Deduct at most 1, overall, for bad forms at infinity	
	Second graph:	
	Axes and both asymptotes + 1 branch	B1
	Other branch	B1
	Deduct at most 1, overall, for bad forms at infinity	
	Choose which is 1st graph and which is 2nd to the benefit of candidates, i.e. best = 1st	
	For differentiating a special case	M1A0A0
	Graphs of special cases	B2 (max) B1 (max)
	If stated in text, turning points need not be labelled on graph	

11 **(i)** $(\mathbf{i} - (2\sin t)\mathbf{j}) \times (4\mathbf{j} - \mathbf{k}) = (2\sin t)\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ M1A1

$PQ = (\mathbf{i} - \mathbf{j}).[(2\sin t)\mathbf{i} + \mathbf{j} + 4\mathbf{k}]/\sqrt{4\sin^2 t + 17}$ dM1A1

$PQ = |1 - 2\sin t| / \sqrt{4\sin^2 t + 17}$ A1

Condone disappearing modulus sign –

Deduct 1 mark if no modulus sign at all

(ii) $PQ = 0 \Rightarrow \sin t = 1/2 \Rightarrow t = \pi/6, 5\pi/6$ or 0.524, 2.62 M1A1 (both)

(iii) Obtains some vector $\perp$ plane BPQ, e.g.,

$(\sqrt{2}\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} - \sqrt{2}\mathbf{j}) = 4\sqrt{2}\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ M1A1

$p(A : BPQ) = (4\sqrt{2}\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}).(\mathbf{i} - \mathbf{j})/\sqrt{57}$ M1A1

$= 4(\sqrt{2} - 1)/\sqrt{57} = 0.219$ A1

or

For **(i)**

$$\ell_1 = r = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$$

$$\ell_2 = r = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2\sin t \\ 0 \end{pmatrix} \qquad \Rightarrow \overrightarrow{PQ} = \begin{pmatrix} \mu - 1 \\ 1 - 4\lambda - 2\mu\sin t \\ \lambda \end{pmatrix}$$

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} = 0 \qquad \Rightarrow 17\lambda + 8\mu\sin t = 4$$

$$PQ \cdot \begin{pmatrix} 1 \\ -2\sin t \\ 0 \end{pmatrix} = 0 \qquad \Rightarrow 8\sin t\lambda + (1 + 4\sin^2 t)\mu = 1 + 2\sin t$$ M1

Whence $\lambda = \dfrac{4(1 - 2\sin t)}{(17 + 4\sin^2 t)}$ and $\mu = \dfrac{17 + 2\sin t}{(17 + 4\sin^2 t)}$ A1 (both)

For $\overrightarrow{PQ}$ $x = \dfrac{2\sin t(1-2\sin t)}{(17+4\sin^2 t)}$ $y = \dfrac{1-2\sin t}{(17+4\sin^2 t)}$ $z = \dfrac{4(1-2\sin t)}{17+4\sin^2 t}$ dM1A1

(λ and μ of the form $\dfrac{a+b\sin t}{17+4\sin^2 t}$)

$$\left|\overrightarrow{PQ}\right| = \sqrt{x^2+y^2+z^2} = \left|\frac{(1-2\sin t)}{(17+4\sin^2 t)}\sqrt{4\sin^2 t+1^2+4^2}\right|$$

$$= \frac{|1-2\sin t|}{\sqrt{17+4\sin^2 t}}$$ A1

For **(iii)** Eliminates λ and μ from parametric equation of plane *BPQ*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -\sqrt{2} \\ 0 \end{pmatrix} + \mu\begin{pmatrix} \sqrt{2} \\ 1 \\ 4 \end{pmatrix}$$ M1A1

to obtain cartesian equation of plane *BPQ*

$4\sqrt{2}x + 4y - 3z = 4(\sqrt{2}-1)$ * A1

Uses distance of point from a line formula with * and (2, 1, 4) M1

d = $\dfrac{8\sqrt{2}+4-12-4\sqrt{2}+4}{\sqrt{32+16+9}} = \dfrac{4(\sqrt{2}-1)}{\sqrt{57}}$

= 0.219 cao A1

12 EITHER

$(1+i\tan\theta)^k = \sec^k\theta(\cos k\theta + i\sin k\theta)$ M1A1

$\sum_{k=0}^{n-1}(1+i\tan\theta)^k = [(1+i\tan\theta)^n - 1]/i\tan\theta$ M1A1

$= \sec^n\theta\sin n\theta\cot\theta + i(\cot\theta - \sec^n\theta\cos n\theta\cot\theta)$ A1

$\Rightarrow \sum_{k=0}^{n-1}\sec^k\theta\cos k\theta = \sec^n\theta\sin n\theta\cot\theta$ (AG) M1A1

Put $\theta = \pi/3$ to obtain second result (AG) M1A1

$x = \cos\theta \Rightarrow \sec\theta = 1/x$, $\cot\theta = x/\sqrt{1-x^2}$ M1A1

Uses these results to obtain final result (AG) M1A1

First 7 marks

$$\sum_0^{n-1}(1+i\tan\theta)^k = \underbrace{\sum_0^{n-1}(\cos k\theta + i\sin k\theta)\sec^k\theta}_{\text{M1A1}} = \underbrace{\frac{\{1-(1+i\tan\theta)^n\}}{-i\tan\theta}}_{\text{M1A1}}$$

$$\therefore \sum_0^{n-1}\cos k\theta\sec^k\theta = \text{Re}\left\{\frac{1-(\cos n\theta + i\sin n\theta)\sec^n\theta}{i\tan\theta}\right\}$$ M1A1

$= \sin n\theta\sec^n\theta\cot\theta$ (AG) A1

12 OR

(i) Reduces $\mathbf{M}_1$ to col echelon form by elementary col operations

or $\mathbf{M}_1^T$ to row echelon form by elementary row operations

$$\mathbf{M}_1^T \to \begin{pmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ hence } \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$ M1

Any correct echelon form A1

Selects 3 li cols from reduced $\mathbf{M}_1$ (or equiv from reduced $\mathbf{M}_1^T$) M1

(Allow M1A1M1 for any other valid method)

Any correct basis of R_1, e.g. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ A1

or for (i)

Reduces $\mathbf{M}_1$ to row echelon form by elementary row operations $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ M1

Any correct row echelon form A1

No working for echelon matrix here, or in **(ii)** gets B1

$rr(\mathbf{M}_1) = 3 \Rightarrow$ any 3 li columns form a basis of $\mathbf{M}_1$ May be implied by answer M1

Any correct basis of R_1, e.g., $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 11 \\ 5 \end{pmatrix} \right\}$ A1

or

det$\mathbf{M}_1 = 0$ (calculator) M1

$\Rightarrow r\mathbf{M}_1 \leq 3$ A1

First 3 columns of $\mathbf{M}_1$ are clearly linearly independent M1

Hence basis A1

(ii) Reduces $\mathbf{M}_2$ to echelon form by elementary row operations $\begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ M1

Any correct row echelon form A1

Valid method to find basis of K_2 M1

(Allow M1A1M1 for any other valid method)

Any correct basis of K_2, e.g., $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ A1

Shows each basis element of K_2 is in R_1 (AG) B1

First 4 marks

$$\mathbf{M}_2^T = \begin{pmatrix} 2 & 5 & 3 & 13 \\ 0 & 1 & -1 & -1 \\ -1 & -3 & -1 & -6 \\ -1 & -3 & -1 & -6 \end{pmatrix} \begin{matrix} \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{matrix} \rightarrow \begin{pmatrix} 2 & 5 & 3 & 13 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{a} \\ \text{b} \\ \text{a}+\text{b}+2\text{c} \\ \text{c}-\text{d} \end{matrix}$$ M1A1

Basis is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ M1A1

(iii) Any valid argument, e.g., W does not contain zero vector so W not a vector space B1

(iv) For any vector $\mathbf{x}$, $\mathbf{M_2M_1x} = \mathbf{M_2}(\alpha\,\mathbf{b_1} + \beta\,\mathbf{b_2} + \gamma\,\mathbf{b_3})$, where $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ are any 3 l.i. basis vectors of R_1, 2 of which must be basis vectors of K_2 M1

Hence if $\mathbf{b_1}$ and $\mathbf{b_2}$ are basis vectors of K_2, then $\mathbf{M_2M_1x} = \gamma\,\mathbf{M_2b_3}$ A1

Hence as dim(range of T_3) = 1, then the dimension of the null space of T_3 = 4 – 1 = 3 A1

or $\mathbf{M_2M_1} = \begin{pmatrix} 0 & -7 & -14 & -14 \\ 0 & -18 & -36 & -36 \\ 0 & -10 & -20 & -20 \\ 0 & -45 & -90 & -90 \end{pmatrix}$ B1

Nullity = 4 – $r(\mathbf{M_2M_1})$ = 3 M1A1