UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1 **May/June 2009**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

1 The equation

$$x^4 - x^3 - 1 = 0$$

has roots α, β, γ, δ. By using the substitution $y = x^3$, or by any other method, find the exact value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. [5]

2 Verify that, for all positive values of n,

$$\frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)} = \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}. \quad [2]$$

For the series

$$\sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)},$$

find

(i) the sum to N terms, [3]

(ii) the sum to infinity. [1]

3 The equation of a curve is $y = \lambda x^2$, where $\lambda > 0$. The region bounded by the curve, the x-axis and the line $x = a$, where $a > 0$, is denoted by R. The y-coordinate of the centroid of R is a. Show that $\lambda = \frac{10}{3a}$. [6]

4 A curve has equation

$$y = \tfrac{1}{3}x^3 + 1.$$

The length of the arc of the curve joining the point where $x = 0$ to the point where $x = 1$ is denoted by s. Show that

$$s = \int_0^1 \sqrt{(1 + x^4)}\,\mathrm{d}x. \quad [2]$$

The surface area generated when this arc is rotated through one complete revolution about the x-axis is denoted by S. Show that

$$S = \tfrac{1}{9}\pi(18s + 2\sqrt{2} - 1). \quad [4]$$

[Do not attempt to evaluate s or S.]

5 Draw a sketch of the curve C whose polar equation is $r = \theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. [2]

On the same diagram draw the line $\theta = \alpha$, where $0 < \alpha < \frac{1}{2}\pi$. [1]

The region bounded by C and the line $\theta = \frac{1}{2}\pi$ is denoted by R. Find the exact value of α for which the line $\theta = \alpha$ divides R into two regions of equal area. [4]

6 A curve has equation

$$(x + y)(x^2 + y^2) = 1.$$

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point (0, 1). [7]

7 Let

$$I_n = \int_0^1 t^n e^{-t}\,dt,$$

where $n \geqslant 0$. Show that, for all $n \geqslant 1$,

$$I_n = nI_{n-1} - e^{-1}. \quad [3]$$

Hence prove by induction that, for all positive integers n,

$$I_n < n!. \quad [5]$$

8 Find the general solution of the differential equation

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 65y = 65x^2 + 8x + 73. \quad [6]$$

Show that, whatever the initial conditions, $\frac{y}{x^2} \to 1$ as $x \to \infty$. [2]

9 The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & -1 \\ 2 & 1 & 5 \end{pmatrix}$$

has eigenvalues 1, 5, 7. Find a set of corresponding eigenvectors. [5]

Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^n = \mathbf{PDP}^{-1}$. [3]
[The evaluation of $\mathbf{P}^{-1}$ is not required.]

Determine the set of values of the real constant k such that $k^n\mathbf{A}^n$ tends to the zero matrix as $n \to \infty$. [3]

10 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equation of each of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases $\lambda > 0$ and $\lambda < 0$. In both cases the coordinates of the turning points must be indicated. [8]

11 The line l_1 is parallel to the vector $4\mathbf{j} - \mathbf{k}$ and passes through the point A whose position vector is $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. The variable line l_2 is parallel to the vector $\mathbf{i} - (2\sin t)\mathbf{j}$, where $0 \leqslant t < 2\pi$, and passes through the point B whose position vector is $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The points P and Q are on l_1 and l_2, respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t. [5]

(ii) Hence find the values of t for which l_1 and l_2 intersect. [2]

(iii) For the case $t = \frac{1}{4}\pi$, find the perpendicular distance from A to the plane BPQ, giving your answer correct to 3 decimal places. [5]

12 Answer only **one** of the following two alternatives.

EITHER

By considering $\sum_{k=0}^{n-1}(1 + \mathrm{i}\tan\theta)^k$, show that

$$\sum_{k=0}^{n-1} \cos k\theta \sec^k \theta = \cot\theta \sin n\theta \sec^n \theta,$$

provided θ is not an integer multiple of $\frac{1}{2}\pi$. [7]

Hence or otherwise show that

$$\sum_{k=0}^{n-1} 2^k \cos\left(\tfrac{1}{3}k\pi\right) = \frac{2^n}{\sqrt{3}} \sin\left(\tfrac{1}{3}n\pi\right). \qquad [2]$$

Given that $0 < x < 1$, show that

$$\sum_{k=0}^{n-1} \frac{\cos(k\cos^{-1}x)}{x^k} = \frac{\sin(n\cos^{-1}x)}{x^{n-1}\sqrt{(1-x^2)}}. \qquad [4]$$

OR

The linear transformations $T_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $T_2 : \mathbb{R}^4 \to \mathbb{R}^4$ are represented by the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively, where

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix}, \qquad \mathbf{M}_2 = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix}.$$

(i) Find a basis for R_1, the range space of T_1. [4]

(ii) Find a basis for K_2, the null space of T_2, and hence show that K_2 is a subspace of R_1. [5]

The set of vectors which belong to R_1 but do not belong to K_2 is denoted by W.

(iii) State whether W is a vector space, justifying your answer. [1]

The linear transformation $T_3 : \mathbb{R}^4 \to \mathbb{R}^4$ is the result of applying T_1 and then T_2, in that order.

(iv) Find the dimension of the null space of T_3. [3]

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