UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2011 question paper

for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/13 Paper 13, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- CIE will not enter into discussions or correspondence in connection with these mark schemes.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Mark	Total
1	Finds four times sum of first *n* squares.	$2^2+4^2+\ldots+(2n)^2=\frac{4n(n+1)(2n+1)}{6}$	M1A1	2	
		$1^2-2^2+3^2-4^2+\ldots-(2n)^2$			
	Subtracts eight times sum of first *n* squares from sum of first 2*n* squares.	$=\frac{2n(2n+1)(4n+1)}{6}-\frac{8n(n+1)(2n+1)}{6}$	M1A1		
	Simplifies.	$=\frac{n(2n+1)}{3}(4n+1-4n-4)=-n(2n+1)$	A1	3	
		Or			
		$\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n-\frac{4n(n+1)(2n+1)}{6}$	(M1A1)		
		$=-2n^2-n$	(A1)		[5]
2	States proposition.	Let P_n be the proposition: $\mathbf{A}=\begin{pmatrix}2&3\\0&1\end{pmatrix}\Rightarrow\mathbf{A}^n=\begin{pmatrix}2^n&3(2^n-1)\\0&1\end{pmatrix}$			
	Shows base case is true.	$\mathbf{A}^1=\begin{pmatrix}2&3\\0&1\end{pmatrix}=\begin{pmatrix}2^1&3\times(2-1)\\0&1\end{pmatrix}\Rightarrow P_1$ is true.	B1		
		Assume P_k is true for some integer *k*.	B1		
	Proves inductive step.	$\mathbf{A}^{k+1}=\begin{pmatrix}2&3\\0&1\end{pmatrix}\begin{pmatrix}2^k&3(2^k-1)\\0&1\end{pmatrix}$	M1		
		$=\begin{pmatrix}2^{k+1}&3.2(2^k-1)+3\\0&1\end{pmatrix}$			
		$=\begin{pmatrix}2^{k+1}&3(2^{k+1}-1)\\0&1\end{pmatrix}$	A1		
	States conclusion.	Since P_1 is true and $P_k\Rightarrow P_{k+1}$, hence by PMI P_n is true $\forall$ positive integers *n*.	A1	5	[5]
3	Uses $\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$	$36=38+2\sum\alpha\beta\Rightarrow\sum\alpha\beta=-1$	M1A1		
	States equation with required roots.	$\therefore t^3+6t^2-t-30=0$ is the required equation.	A1	3	
	Factorises	$\Rightarrow(t-2)(t+3)(t+5)=0$	M1A1		
	Gives values of α, β, γ.	Hence α, β, and γ are 2, –3 and –5 (in any order). N.B. Answers written down with no working get B1.	A1	3	[6]

Qu No	Commentary	Solution	Marks	Part Mark	Total
4	Differentiates with respect to x.	$2y^2 + 4xyy' + 6xy + 3x^2 y' = 0$	B1B1		
	Substitutes (–1, 1)	$2 - 4y' - 6 + 3y' = 0 \Rightarrow y' = -4$ (AG)	B1	3	
	Differentiates again.	$4yy' + (4y + 4xy')y' + 4xyy'' + 6y + 6xy'$	B1B1		
		$+ 6xy' + 3x^2 y'' = 0$	B1		
	Substitutes (–1, 1) and $y' = -4$	$-16 - 80 - 4y'' + 6 + 24 + 24 + 3y'' = 0$	M1		
		$\Rightarrow y'' = -42$	A1	5	[8]
5	Uses $\tan^2 x = \sec^2 x - 1$	$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \mathrm{d}x = \int_0^{\frac{\pi}{4}} \tan^{n-2} x(\sec^2 x - 1)\mathrm{d}x$	M1		
	Integrates	$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \mathrm{d}x - I_{n-2} = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$	M1A1		
		Or			
		$I_n = \left[\tan^{n-1} x\right]_0^{\frac{\pi}{4}}$			
		$- \int_0^{\frac{\pi}{4}} (n-2)\tan^{n-3} x \sec^2 x \tan x \mathrm{d}x - I_{n-2}$	(M1)		
		$= 1 + (n-2)\int_0^{\frac{\pi}{4}} \tan^{n-2} x(1 + \tan^2 x)\mathrm{d}x - I_{n-2}$	(A1)		
	Obtains reduction formula.	$= \frac{1}{n-1} - I_{n-2}$ (AG)	A1	4	
	Evaluates I_0	$I_0 = \int_0^{\frac{\pi}{4}} 1\mathrm{d}x = \frac{\pi}{4}$	B1		
		$I_2 = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\mathrm{d}x = \left[\tan x - x\right]_0^{\frac{\pi}{4}} = 1 - \frac{\pi}{4}$	(B1)		
	Uses reduction formula.	$I_2 = 1 - I_0 \qquad I_4 = \frac{1}{3} - 1 + I_0$	M1A1		
		$I_6 = \frac{1}{5} - \frac{1}{3} + 1 - I_0 \quad I_8 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4}$ (AG)	A1	4	[8]

Qu No	Commentary	Solution	Marks	Part Mark	Total
5	**Alternative for first part:**	$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{n-1}x) = (n-1)\tan^{n-2}x\sec^2 x$			
		$= (n-1)\tan^{n-2}x(1+\tan^2 x)$	M1		
		$= (n-1)\tan^{n-2}x + (n-1)\tan^n x$	A1		
		Integrating with respect to x, between 0 and $\frac{\pi}{4}$			
		$\left[\tan^{n-1}x\right]_0^{\frac{\pi}{4}} = (n-1)I_{n-2} + (n-1)$	M1		
		$\Rightarrow 1 = (n-1)I_{n-2} + (n-1)I_n$			
		$\Rightarrow I_n = \frac{1}{n-1} - I_{n-2}$	A1	4	
6	Sketches each curve on same diagram.	Sketch of C_1 (relevant part only required). Sketch of C_2 (generous on tangency features).	B1 B2	3	
	States the value of β.	$\beta = \frac{\pi}{6}$.	B1	1	
	Adds $\frac{1}{12}$ of area of circle to sector of C_2 from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$.	$\frac{1}{12}\pi a^2 + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4a^2\cos^2 2\theta\,\mathrm{d}\theta$	B1M1		
	Uses double angle formula and integrates.	$= \frac{1}{12}\pi a^2 + a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cos 4\theta + 1)\,\mathrm{d}\theta$			
		$= \frac{1}{12}\pi a^2 + a^2\left[\frac{\sin 4\theta}{4} + \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$	M1		
	Obtains printed result.	$= a^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$ (AG)	A1	4	**[8]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
7	Differentiates	$\dot{x} = e^t(\cos t - \sin t) \quad \dot{y} = e^t(\sin t + \cos t)$	B1		
		$\dot{s} = \sqrt{e^{2t}(1 - 2\sin t\cos t + 1 + 2\sin t\cos t)} = \sqrt{2}e^t$	B1		
	Uses arc length formula	$s = \sqrt{2}\int_0^\pi e^t \, dt$	M1		
		$= \sqrt{2}(e^\pi - 1) \quad (= 31.3)$	A1	4	
	Uses surface area formula and obtains correct integral.	$S = 2\pi\int_0^\pi e^t \sin t\sqrt{2}e^t dt = 2\sqrt{2}\pi\int_0^\pi e^{2t}\sin t dt$	M1A1		
		Let $I = \int e^{2t}\sin t dt$			
	Integrates by parts twice.	$= -e^{2t}\cos t + \int 2e^{2t}\cos t dt$	M1		
		$-e^{2t}\cos t + 2e^{2t}\sin t - \int 4e^{2t}\sin t dt$	A1		
	Sees original again.	$5I = 2e^{2t}\sin t - e^{2t}\cos t$	M1		
		$\Rightarrow I = \frac{e^{2t}}{5}(2\sin t - \cos t)$	A1		
	Obtains surface area.	$S = 2\sqrt{2}\pi\left[\frac{e^{2t}}{5}(2\sin t - \cos t)\right]_0^\pi = \frac{2\sqrt{2}\pi}{5}(e^{2\pi} + 1)$	A1	7	
		(= 953) (N.B. If 953 written down with no working award B1 in place of the final 5 marks.)			
	Alternative method for integrating $\int e^{2t}\sin t \, dt$.	$\text{Im}\left\{\int e^{2t} \cdot e^{it} \, dt\right\} = \text{Im}\left\{\int e^{(2+i)t} \, dt\right\}$	M1		
		$= \text{Im}\left[\frac{e^{(2+i)t}}{2+i}\right] = \text{Im}\left[\frac{e^{2t}}{5}(\cos t + i\sin t)(2 - i)\right]$	A1M1		
		$= \frac{1}{5}e^{2t}(2\sin t - \cos t)$	A1		[11]

Qu No	Commentary	Solution	Marks	Part Mark	Total
8	Forms and solves AQE.	$m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2\text{i}$	M1		
	States CF	CF $\text{e}^{-t}(A\cos 2t + B\sin 2t)$ (OE)	A1		
	States form for PI.	PI $x = p\cos t + q\sin t \Rightarrow \dot{x} = -p\sin t + q\cos t$ $\Rightarrow \ddot{x} = -p\cos t - q\sin t$	M1		
	Substitutes in equation.	$-p\cos t - q\sin t - 2p\sin t + 2q\cos t$ $+5p\cos t + 5q\sin t = 10\sin t$			
	Obtains values for p and q by comparing coefficients.	$4p + 2q = 0$ and $-2p + 4q = 10$	M1		
		$\Rightarrow p = -1, \quad q = 2$	A1		
	States GS.	GS $x = \text{e}^{-t}(A\cos 2t + B\sin 2t) + 2\sin t - \cos t$ (OE)	A1	6	
	Uses initial conditions to evaluate constants.	$t = 0 \quad x = 5 \quad \Rightarrow \quad A = 6$	B1		
		$\dot{x} = -\text{e}^{-t}(A\cos 2t + B\sin 2t)$ $+\text{e}^{-t}(-2A\sin 2t + 2B\cos 2t) + 2\cos t + \sin t$	M1		
		$2 = -6 + 2B + 2 \Rightarrow B = 3$	A1		
	States particular solution.	$x = \text{e}^{-t}(6\cos 2t + 3\sin 2t) + 2\sin t - \cos t$ (OE)	A1	4	
	Gives req. approximate solution.	As $t \to \infty \quad x \approx 2\sin t - \cos t$ (The final mark is independent of A and B).	B1	1	[11]
9 (i)	States vertical asymptote.	$x = 1$	B1	1	
(ii)	States the value of a.	$a = 2$	B1		
		$y = ax + a + b + \dfrac{a+b+c}{(x-1)}$	M1		
	Divides. Compares coefficients to obtain b.	$2 + b = -5 \Rightarrow b = -7$ (AG)	A1	3	
		Or			
		$y = 2x - 5 + \dfrac{a}{x-1}$	(M1)		
		$= \dfrac{2x^2 - 7x + 5 + a}{x-1}$	(B1A1)		
		Equate coefficients to obtain			
		$a = 2, \quad b = -7$			

Qu No	Commentary	Solution	Marks	Part Mark	Total
(iii)	Differentiates and uses given value of x to obtain c.	$y' = 2 - \frac{(c-5)}{(x-1)^2} = 0$	M1A1		
		When $x = 2$ then $c = 7$	A1	3	
(iv)	Forms quadratic in x.	Let $y = \frac{2x^2 - 7x + 7}{(x-1)} = k$			
		$\Rightarrow 2x^2 - (7+k)x + 7 + k = 0$	B1		
	Uses discriminant.	No real roots $\Rightarrow (7+k)^2 - 8(7+k) < 0$	M1		
		$\Rightarrow k^2 + 6k - 7 < 0$	A1		
		$\Rightarrow (k+7)(k-1) < 0$			
	Obtains required result.	$\Rightarrow -7 < k < 1$	A1	4	**[11]**
10	Uses vector product to find normal to plane.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & 6 & 1 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$	M1A1		
	Uses **r.n =** constant. Obtains cartesian equation of plane.	Equation of plane: $5x - 3y - 2z =$ constant $30 - 15 - 8 = 7$ $5x - 3y - 2z = 7$	M1 A1	4	
		Alternatively:			
		$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$	(M1)		
		$x = 6 + \lambda + 4\mu$ $y = 5 + \lambda + 6\mu$ $z = 4 + \lambda + \mu$	(A1)		
		Eliminates λ and μ.	(M1)		
		Obtains $5x - 3y - 2z = 7$	(A1)		

Qu No	Commentary	Solution	Marks	Part Mark	Total
10 Contd.	Finds equation of perpendicular to plane through given point.	Equation of perpendicular: $\mathbf{r} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k} + t(5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$	M1		
	Finds value of parameter at point in plane. Obtains foot of perpendicular.	$5(1+5t) - 3(10-3t) - 2(3-2t) = 7$ $\Rightarrow t = 1$ Foot of perpendicular is $6\mathbf{i} + 7\mathbf{j} + \mathbf{k}$.	M1 A1 A1	4	
	Alternatively: Form sufficient equations, using orthogonality. Two will suffice if foot of perpendicular is expressed using parametric equation of plane.	Let foot of perpendicular be $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and using othogonality: $\begin{pmatrix} a-1 \\ b-10 \\ c-3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow a+b+c = 14$ $\begin{pmatrix} a-1 \\ b-10 \\ c-3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix} = 0 \Rightarrow 4a+6b+c = 67$			
		$a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ lies in plane of l_1 and l_2: $5a - 3b - 2c = 7$	(M1A1)		
		$\Rightarrow 6\mathbf{i} + 7\mathbf{j} + \mathbf{k}$	(M1A1)		
	Finds direction of common perpendicular.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$	M1A1		
	Forms vector between known points on l_1 and l_3.	$\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$			
	Finds shortest distance by projection.	$\frac{1}{\sqrt{16+1+25}} \left\| \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \right\| = \frac{10}{\sqrt{42}}$ $(= 1.54)$	M1A1 A1	5	**[13]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
10 Contd.	**Alternative for last part:**	Let P be on l_1 and Q be on $l_{3.}$ $\mathbf{p} = \begin{pmatrix} 6+\lambda \\ 5+\lambda \\ 4+\lambda \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1+2v \\ 10-3v \\ 3+v \end{pmatrix}$			
		$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} -5-\lambda+2v \\ 5-\lambda-3v \\ -1-\lambda-3v \end{pmatrix}$	(M1)		
		Uses orthogonality conditions: $\Rightarrow \overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow -1-3\lambda = 0 \Rightarrow \lambda = -\frac{1}{3}$	(M1)		
		$\overrightarrow{PQ} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \Rightarrow -26+14v = 0 \Rightarrow v = \frac{13}{7}$	(A1)		
		$\Rightarrow \overrightarrow{PQ} = \frac{1}{21}\begin{pmatrix} -20 \\ -5 \\ 25 \end{pmatrix}$	(A1)		
		$\Rightarrow \left\|\overrightarrow{PQ}\right\| = \frac{5}{21}\sqrt{4^2+1^2+5^2} = \frac{5}{21}\sqrt{42}$	(A1)	(5)	

Qu No	Commentary	Solution	Marks	Part Mark	Total
11	**EITHER**				
(i)	Writes **P** and **D**. (Note: Columns can be in any order, but must match.)	$\mathbf{P} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	B1B1		
	Finds Det **P.**	Det **P** = 2	B1		
	Finds inverse of **P**. (Adj ÷ Det)	$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (No working 1/3)	M1A1		
		Row operations M1A1A1 (3 errors).			
	Finds expression for **A.**	$\mathbf{A} = \mathbf{PDP}^{-1}$	M1		
	Evaluates **A.**	$\mathbf{A} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} . \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	M1A1		
		$= \begin{pmatrix} 1.5 & 0.5 & 0.5 \\ 1.5 & 0.5 & 1.5 \\ -1 & 1 & 0 \end{pmatrix}$	A1	9	
(ii)	Finds expression for $\mathbf{A}^{2n}$	$\mathbf{A}^{2n} = \mathbf{PD}^{2n}\mathbf{P}^{-1}$	M1		
		$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2n} \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	A1		
		$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 2^{2n} \\ 1 & 0 & 2^{2n} \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	M1A1		
	Evaluates.	$= \frac{1}{2}\begin{pmatrix} 2^{2n}+1 & 2^{2n}-1 & 2^{2n}-1 \\ 2^{2n}-1 & 2^{2n}+1 & 2^{2n}-1 \\ 0 & 0 & 2 \end{pmatrix}$	A1	5	[14]

Qu No	Commentary	Solution	Marks	Part Mark	Total
	EITHER (Alternative)				
(i)	Uses **Ae** = λ**e** (3 times)	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\quad b-c=0,\ e-f=-1,\ h-j=1$			
	Forms 3 linear equations (3 times)	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad -a+c=-1,\ -d+f=0,\ -g+j=1$	M1A1		
		$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ $\quad a+b=2,\ d+e=2,\ g+h=0$			
	Solves one set of equations. Solves other two sets.		M1		
	Writes **A.**	$\mathbf{A} = \begin{pmatrix} 1.5 & 0.5 & 0.5 \\ 1.5 & 0.5 & 1.5 \\ -1 & 1 & 0 \end{pmatrix}$	A1	4	
(ii)	Writes **P** and **D**.	$\mathbf{P} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	B1B1		
	Finds inverse of **P.**	$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	B1 M1A1	5	
	Finds $\mathbf{A}^{2n}$.	$\mathbf{A}^{2n} = \mathbf{PD}^{2n}\mathbf{P}^{-1}$	M1		
		$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2n} \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	A1		
		$= \frac{1}{2}\begin{pmatrix} 0 & -1 & 2^{2n} \\ 1 & 0 & 2^{2n} \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	M1A1		
		$= \frac{1}{2}\begin{pmatrix} 2^{2n}+1 & 2^{2n}-1 & 2^{2n}-1 \\ 2^{2n}-1 & 2^{2n}+1 & 2^{2n}-1 \\ 0 & 0 & 2 \end{pmatrix}$	A1	5	**[14]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
11	**OR**				
	Reduces matrix to echelon form.	$\begin{pmatrix}1 & -1 & -1 & 1\\ 2 & -1 & -4 & 3\\ 3 & -3 & -2 & 2\\ 5 & -4 & -6 & 5\end{pmatrix} \to \dots \begin{pmatrix}1 & -1 & -1 & 1\\ 0 & 1 & -2 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$	M1A1		
	Obtains rank.	$r(\mathbf{A}) = 4 - 1 = 3$	A1	3	
	Use system of equations, or any other method (see below).	$x - y - z + t = 0$ $y - 2z + t = 0$ $z - t = 0$	M1		
		$\Rightarrow t = \lambda,\ z = \lambda,\ y = \lambda,\ x = \lambda$			
	Finds basis of null space.	$\therefore$ Basis of null space is $\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$	A1		
	Obtains general solution.	$\left\{\mathbf{Ax} - \begin{pmatrix}p\\q\\r\\0\end{pmatrix}\right\} = \mathbf{0} \Rightarrow \mathbf{x} = \begin{pmatrix}p\\q\\r\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ (AG)	M1A1	4	
	Finds values of p, q and r, e.g. by solving a set of equations.	$p - q - r = 3$ $2p - q - 4r = 7$ $3p - 3q - 2r = 8$	M1		
	Award B2 (all correct) or B1 (two correct), with no working.	$p = 1,\quad q = -1 \quad r = -1$	A1, A1	3	
	Solves for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{11}{4}$.	$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{11}{4} \Rightarrow 4\lambda^2 - 2\lambda + \frac{1}{4} = 0$	M1A1		
		$\Rightarrow \left(2\lambda - \frac{1}{2}\right)^2 = 0 \Rightarrow \lambda = \frac{1}{4} \Rightarrow \mathbf{x} = \begin{pmatrix}1.25\\-0.75\\-0.75\\0.25\end{pmatrix}$	M1A1	4	**[14]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
11 Contd.	Alternative methods for 2nd part.	Writes $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and forms equations from			
		$\mathbf{Ax} = p\begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} + q\begin{pmatrix} -1 \\ -1 \\ -3 \\ -4 \end{pmatrix} + r\begin{pmatrix} -1 \\ -4 \\ -2 \\ -6 \end{pmatrix}$	M1A1		
		$x_1 - x_2 - x_3 + x_4 = p - q - r$ $2x_1 - x_2 - 4x_3 + 3x_4 = 2p - q - 4r$ $3x_1 - 3x_2 - 2x_3 + 2x_4 = 3p - 3q - 2r$ $5x_1 - 4x_2 - 6x_3 + 5x_4 = 5p - 4q - 6r$	M1		
		Obtains, for example, $x_1 = x_4 + p$ $x_2 = x_4 + q$ $x_3 = x_4 + r$			
		Sets $x_4 = \lambda$ to obtain:	A1		
		$\mathbf{x} = \begin{pmatrix} p+\lambda \\ q+\lambda \\ r+\lambda \\ \lambda \end{pmatrix}$			
		Mark similarly if equations obtained from reduced augmented matrix. Those who work in reverse direction and merely verify the result get M1A1 i.e. 2/4.			