

**CAMBRIDGE INTERNATIONAL EXAMINATIONS**  
**General Certificate of Education Advanced Level**

**FURTHER MATHEMATICS**  
**PAPER 1**

**9231/1**

**OCTOBER/NOVEMBER SESSION 2002**

3 hours

Additional materials:  
Answer paper  
Graph paper  
List of Formulae (MF10)

**TIME** 3 hours

**INSTRUCTIONS TO CANDIDATES**

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION FOR CANDIDATES**

The number of marks is given in brackets [ ] at the end of each question or part question.

The use of a calculator is expected, where appropriate.

Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.

You are reminded of the need for clear presentation in your answers.

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**This question paper consists of 5 printed pages and 3 blank pages.**

1 Given that

$$u_n = e^{nx} - e^{(n+1)x},$$

find  $\sum_{n=1}^N u_n$  in terms of  $N$  and  $x$ . [2]

Hence determine the set of values of  $x$  for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity for cases where this exists. [3]

2 The equation

$$x^4 + x^3 + Ax^2 + 4x - 2 = 0,$$

where  $A$  is a constant, has roots  $\alpha, \beta, \gamma, \delta$ . Find a polynomial equation whose roots are

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}. \quad [2]$$

Given that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2},$$

find the value of  $A$ . [3]

3 It is given that, for  $n = 0, 1, 2, 3, \dots$ ,

$$a_n = 17^{2n} + 3(9)^n + 20.$$

Simplify  $a_{n+1} - a_n$ , and hence prove by induction that  $a_n$  is divisible by 24 for all  $n \geq 0$ . [6]

4 It is given that, for  $n \geq 0$ ,

$$I_n = \int_0^1 x^n e^{-x^2} dx.$$

(i) Find  $I_1$  in terms of  $e$ . [1]

(ii) Show that

$$I_{n+2} = \frac{n+1}{2} I_n - \frac{1}{2e}. \quad [3]$$

(iii) Find  $I_5$  in terms of  $e$ . [3]

5 The curve  $C$  has polar equation  $r\theta = 1$ , for  $0 < \theta \leq 2\pi$ .

(i) Use the fact that  $\frac{\sin \theta}{\theta}$  tends to 1 as  $\theta$  tends to 0 to show that the line with cartesian equation  $y = 1$  is an asymptote to  $C$ . [2]

(ii) Sketch  $C$ . [1]

The points  $P$  and  $Q$  on  $C$  correspond to  $\theta = \frac{1}{6}\pi$  and  $\theta = \frac{1}{3}\pi$  respectively.

(iii) Find the area of the sector  $OPQ$ , where  $O$  is the origin. [3]

(iv) Show that the length of the arc  $PQ$  is

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{1 + \theta^2}}{\theta^2} d\theta. \quad [2]$$

6 A curve has equation  $x^3 + xy^2 - y^3 = 3$ .

(i) Show that there is no point of the curve at which  $\frac{dy}{dx} = 0$ . [4]

(ii) Find the values of  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at the point  $(1, -1)$ . [5]

7 Given that  $z = \cos \theta + i \sin \theta$ , show that

(i)  $z - \frac{1}{z} = 2i \sin \theta$ , [1]

(ii)  $z^n + z^{-n} = 2 \cos n\theta$ . [2]

Hence show that

$$\sin^6 \theta = \frac{1}{32}(10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta). \quad [3]$$

Find a similar expression for  $\cos^6 \theta$ , and hence express  $\cos^6 \theta - \sin^6 \theta$  in the form  $a \cos 2\theta + b \cos 6\theta$ . [3]

8 The value of the assets of a large commercial organisation at time  $t$ , measured in years, is  $\$(10^8 y + 10^9)$ . The variables  $y$  and  $t$  are related by the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 15 \cos 3t - 3 \sin 3t.$$

Find  $y$  in terms of  $t$ , given that  $y = 3$  and  $\frac{dy}{dt} = -2$  when  $t = 0$ . [9]

Show that, for large values of  $t$ , the value of the assets is less than  $\$9.5 \times 10^8$  for about a third of the time. [3]

- 9 The planes  $\Pi_1$  and  $\Pi_2$ , which meet in the line  $l$ , have vector equations

$$\begin{aligned} \mathbf{r} &= 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_1(2\mathbf{i} + 3\mathbf{k}) + \phi_1(-4\mathbf{j} + 5\mathbf{k}), \\ \mathbf{r} &= 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_2(3\mathbf{j} + \mathbf{k}) + \phi_2(-\mathbf{i} + \mathbf{j} + 2\mathbf{k}), \end{aligned}$$

respectively. Find a vector equation of the line  $l$  in the form  $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ . [5]

Find a vector equation of the plane  $\Pi_3$  which contains  $l$  and which passes through the point with position vector  $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ . Find also the equation of  $\Pi_3$  in the form  $ax + by + cz = d$ . [4]

Deduce, or prove otherwise, that the system of equations

$$\begin{aligned} 6x - 5y - 4z &= -32, \\ 5x - y + 3z &= 24, \\ 9x - 2y + 5z &= 40, \end{aligned}$$

has an infinite number of solutions. [3]

- 10 The linear transformation  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is represented by the matrix  $\mathbf{H}$ , where

$$\mathbf{H} = \begin{pmatrix} 1 & 2 & -3 & -5 \\ -1 & 4 & 5 & 1 \\ 2 & 3 & 0 & -3 \\ -3 & 5 & 7 & 2 \end{pmatrix}.$$

(i) Find the dimension of the range space of  $T$ . [3]

(ii) Find a basis for the null space of  $T$ . [3]

(iii) It is given that  $\mathbf{x}$  satisfies the equation

$$\mathbf{H}\mathbf{x} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix}.$$

Using the fact that

$$\mathbf{H} \begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix},$$

find the least possible value of  $|\mathbf{x}|$ . [7]

[For the vector  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ ,  $|\mathbf{x}| = \sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)}$ .]

11 Answer only **one** of the following two alternatives.

**EITHER**

The vector  $\mathbf{e}$  is an eigenvector of the square matrix  $\mathbf{G}$ . Show that

- (i)  $\mathbf{e}$  is an eigenvector of  $\mathbf{G} + k\mathbf{I}$ , where  $k$  is a scalar and  $\mathbf{I}$  is an identity matrix,
- (ii)  $\mathbf{e}$  is an eigenvector of  $\mathbf{G}^2$ .

[5]

Find the eigenvalues, and corresponding eigenvectors, of the matrices  $\mathbf{A}$  and  $\mathbf{B}^2$ , where

$$\mathbf{A} = \begin{pmatrix} 3 & -3 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -5 & -3 & 0 \\ 1 & -8 & 1 \\ -1 & 3 & -6 \end{pmatrix}. \quad [9]$$

**OR**

The curve  $C$  has equation

$$y = \frac{(x-a)(x-b)}{x-c},$$

where  $a, b, c$  are constants, and it is given that  $0 < a < b < c$ .

- (i) Express  $y$  in the form

$$x + P + \frac{Q}{x-c},$$

giving the constants  $P$  and  $Q$  in terms of  $a, b$  and  $c$ . [3]

- (ii) Find the equations of the asymptotes of  $C$ . [2]

- (iii) Show that  $C$  has two stationary points. [5]

- (iv) Given also that  $a + b > c$ , sketch  $C$ , showing the asymptotes and the coordinates of the points of intersection of  $C$  with the axes. [4]

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