FURTHER MATHEMATICS

Paper 9231/01
Paper 1

General comments

Only the occasional script of outstanding quality and a small number of good quality scripts were received in response to this examination. There were a significant number of very poor scripts. The better candidates presented their work well, while the work of weaker candidates was untidy with many incomplete or deleted attempts at solutions. Many candidates were unable to differentiate or integrate accurately and algebraic technique was often weak. Numerical answers were frequently inaccurate.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. The vast majority of candidates made substantial attempts at nearly all the questions. Once again there were very few misreads and this year there were few rubric infringements. The occasional candidate, who could not fully attempt either of the two alternatives in the final question, would submit two partial solutions.

It was felt that candidates had been well prepared for all parts of the syllabus and had a sound knowledge of certain areas. Induction, linear spaces and complex numbers, however, still remained problematical for many candidates. Recent improvements in vector work seem to have been maintained.

Comments on specific questions

Question 1

Many candidates were able to gain the first mark by producing the result $S = 2\pi a^2 \int_0^{\sqrt{2}} t\sqrt{4t^2+1}\,\mathrm{d}t$. The number of candidates who could successfully perform the integration, either directly, or using a suitable substitution, was disappointingly small. The fact that the answer was given in the question meant that full working was required.

Question 2

There were many complete and accurate answers to this question. Almost all candidates were able to establish the result $\frac{2n+3}{n(n+1)} = \frac{3}{n} - \frac{1}{n+1}$. Many were then able to use the method of differences to show that $\sum_{n=1}^{N} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1} = \frac{1}{3} - \frac{1}{(N+1)}\left(\frac{1}{3}\right)^{N+1}$, from which they were able to deduce the sum to infinity correctly.

Answers: $\frac{1}{3} - \frac{1}{3^{N+1}(N+1)}$, $\frac{1}{3}$.

Question 3

This proved to be a most difficult question for a large number of candidates.

Often only one or two marks were gained for stating inductive hypothesis and/or demonstrating that the result was true for $n = 1$.

There seemed to be much confusion in the minds of candidates over what constituted a polynomial in x of degree n, so few were able to show that H_k is true $\Rightarrow$ H_{k+1} is true.

The most concise solutions stated that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}P_k(x)\right) = 2x\mathrm{e}^{x^2}P_k(x) + \mathrm{e}^{x^2}P'(x)$ and explained that the first term was the product of e^{x^2} and a polynomial in x of degree $k + 1$, while the second term was the product of e^{x^2} and a polynomial in x of degree $k - 1$, thus producing $\mathrm{e}^{x^2}P_{k+1}(x)$. Occasionally a candidate, who did get to this stage, did not write a full conclusion and so dropped the final mark.

Question 4

The proof of the first result was done well by many candidates. There were two common methods, both of which required forming a pair of simultaneous equations.

Firstly: α is a root of the equation $\alpha^3 - 8\alpha^2 + 5 = 0$ hence $\alpha^2(\alpha - 8) + 5 = 0$.
Also $\alpha + \beta + \gamma = 8$ hence $\alpha - 8 = -(\beta + \gamma)$. The result is then obvious.

Secondly: $\alpha\beta + \beta\gamma + \gamma\alpha = 0$ hence $\alpha(\beta + \gamma) = -\beta\gamma$.
Also $\alpha\beta\gamma = -5$ hence $\beta\gamma = -\frac{5}{\alpha}$. The result is again obvious.

Two strategies were evident in the second part of the question, but fewer candidates were able to complete this part of the question. Since the product of the roots is negative, there are one or three negative roots. Call a negative root α. Using the first result $(\beta + \gamma)$ must be positive and from the product of the roots $\beta\gamma$ must be positive also, so both roots β and γ must be positive. Alternatively, as before, there must be one or three negative roots, but because the sum of the roots is positive, it is impossible to have 3 negative roots, so the only possibility is one negative root and two positive roots.

Question 5

There were many good solutions to this question, a large number of which scored full marks. The most successful candidates usually rewrote the given equation as $y = x^2 + 2\ln x + 2\ln y$. Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + \frac{2}{x} + \frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 - \frac{2}{x^2} - \frac{2}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{2}{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. It is then straightforward to substitute $x = 1$ and $y = 1$ to obtain the correct results.

Answers: -4, 32.

Question 6

This question was done well by a large number of candidates. Using vector products to find the normals to the two planes was well known by nearly all candidates. The great majority were then able to use the scalar product to find the angle between these two normals, which is the angle between the planes.

Answer: $16.1°$ or 0.280 radians.

Question 7

Most candidates drew a reasonable sketch. Some lost a mark by making their curve symmetrical about $\theta = \frac{\pi}{2}$. Most were able to write down the correct integral $\frac{1}{2}\int_0^{\pi} \theta^2 \sin^2\theta \, d\theta$ for the area. Replacing $\sin^2\theta$ by $\frac{1}{2}(1-\cos 2\theta)$ defeated a disappointingly large number. It then remained necessary to integrate by parts twice, taking care over the signs, which caused problems for a good number of candidates who got this far.

Answer: $\frac{\pi^3}{12} - \frac{\pi}{8}$ in any correct form.

Question 8

(i) Almost all candidates obtained $(e^x + e^{-x})^n + (n-1)(e^x + e^{-x})^{n-2}(e^x - e^{-x})^2$. Very few knew that $(e^x - e^{-x})^2 = (e^x + e^{-x})^2 - 4$.

(ii) Without having completed part **(i)** many candidates could reverse the printed result and obtain the reduction formula.

(iii) There was much good work on this final part and most knew that they required $\frac{I_4}{2I_2}$. They were then able to use the reduction formula twice and also find $I_0 = \ln 2$. Some poor algebra and arithmetic meant that some did not find the correct final value.

Answer: **(iii)** 2.398.

Question 9

A very high proportion of candidates were only able to give the five fifth roots as $e^{2k\pi i/5}$ for k = 0, 1, 2, 3, 4.

Those who were able to go further used the factorisation $z^5 - 1 = 0 \Rightarrow (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$. Hence if z is a complex root $z^4 + z^3 + z^2 + z + 1 = 0$. Writing $z = w - 1$, $(w-1)^4 + (w-1)^3 + (w-1)^2 + w = 0$ and its roots are $1 + e^{2k\pi i/5}$ for k = 1, 2, 3, 4. A careful sketch reveals that the two with the smaller modulus are when k = 2, 3. Application of the appropriate trigonometric formulae gives the amplitudes.

Answers: $1 + e^{2k\pi i/5}$ for k = 1, 2, 3, 4; $\pm\frac{2\pi}{5}$.

Question 10

This question was the least well-done question on the paper. A comment, with some evidence, to the effect that $V_1 \cup V_2$ was not closed under addition was required for part **(i)**. In part **(ii)** identifying ($\mathbf{b}_1$, $\mathbf{b}_2$) as a basis gave the dimension of $V_1 \cap V_2$ as 2. Showing V_3 is closed under addition was sufficient for the next mark. Identifying ($\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_4$) as a basis and showing the dimension of V_3 to be 3 gained the next two marks.

The final part could be tackled as follows.

$$\begin{pmatrix}4\\4\\2\\5\end{pmatrix} = 2\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - 3\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + 5\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \Rightarrow \begin{pmatrix}4\\4\\2\\5\end{pmatrix} \in V_3$$

$$\begin{pmatrix}5\\4\\2\\5\end{pmatrix} = \begin{pmatrix}4\\4\\2\\5\end{pmatrix} + \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \notin V_3 \;\Rightarrow \begin{pmatrix}5\\4\\2\\5\end{pmatrix} \notin V_3 \text{ or } \begin{pmatrix}x\\y\\z\\t\end{pmatrix} \in V_3 \Rightarrow x = y \text{ not true for } \begin{pmatrix}5\\4\\2\\5\end{pmatrix}.$$

Answers: **(i)** Not closed under addition; **(ii)** 2, ($\mathbf{b}_2$,$\mathbf{b}_3$).

Question 11

There were many good answers to large parts of this question. Most candidates were able to obtain the correct characteristic equation $\lambda^3 - \lambda^2 - 9\lambda + 9 = 0$ and hence obtain the correct eigenvalues. Eigenvectors were usually obtained from a set of linear equations. The vector product method might have proved to be more accurate for some candidates, whose algebra was weak. The eigenvalues of **B** were $(-3 - k)$, $(1 - k)$ and $(3 - k)$. **P** and **D** could then be written down.

Answers: $-3, 1, 3;\ \begin{pmatrix}-17\\6\\7\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}; \begin{pmatrix}-17 & 1 & 1\\6 & -2 & 0\\7 & 1 & 1\end{pmatrix}; \begin{pmatrix}-(3+k)^3 & 0 & 0\\0 & (1-k)^3 & 0\\0 & 0 & (3-k)^3\end{pmatrix}.$

Question 12 EITHER

(i) Dividing and comparing coefficients enabled most candidates to obtain values for a and b.

(ii) Differentiating and putting $x = -1$ enabled most candidates to find the value of c.

(iii) Very few candidates knew how to use the discriminant to find the impossible values for y. Some successfully used the turning points and their graph to do this part of the question.

(iv) Graphs suffered from an inaccurate oblique asymptote and inaccurate turning points; consequently few candidates gained marks here.

Answers: **(i)** 2, 3; **(ii)** -2; **(iii)** $-25 < y < -1$.

Question 12 OR

Many candidates struggled to find correct expressions for y' and y'' in terms of w, so only the better candidates were able to score the first four marks. Most doing this question were able to accurately find the complementary function, particular integral and general solution for the differential equation. Hardly any candidates wrote down an expression for y in terms of x, so they were unable to consider the effect on e^{-x} of x tending to infinity and thus being able to deduce an expression for f(x).

Answers: $w = e^{-x}(A\cos(2x) + B\sin(2x))$ and $f(x) = -\dfrac{1}{x^2}$ or similar expression.

FURTHER MATHEMATICS

Paper 9231/02
Paper 2

General comments

The quality of the candidates' work varied very greatly so in that sense the paper discriminated well. It did not appear to be too long for the time allowed, since the better candidates were able to attempt all questions. In general the Statistics questions were answered more successfully, and in the single question offering a choice, namely **Question 11**, the Statistics option attracted more attempts. Although as usual some questions were found by the candidates to be more demanding than others, none appeared to be unduly hard or easy.

Comments on specific questions

Question 1

Despite the required moment of inertia of *AOB* being given, the majority of candidates were unable to produce a valid derivation of it. The starting point is to apply the expression given in the *List of Formulae* for the moment of inertia of a rectangular lamina about the perpendicular axis through its centre to *ABCD*. A common error here was to use wrong values for the length of the sides. The perpendicular axis theorem then gives the moment of inertia of the lamina about *DB*, and the result for the triangle *AOB* is one-quarter of this. The second part requires that the product of the moment of inertia and the angular acceleration be equated to the moment of the mass about *OB*, and the linear acceleration then follows by multiplying the angular acceleration by the distance *a*. This last step was frequently omitted, while other candidates did not include *g* in their expression for the moment.

Answers: $\frac{4ma^2}{3}$; $2g$.

Question 2

The SHM parameter ω is most easily found from the standard formula $v^2 = \omega^2(a^2 - x^2)$ with here $v = 4$, $a = 2.5$ and $x = 0.5$, but a many candidates used incorrect values for a and/or x. The number of oscillations per minute is then $\frac{60\omega}{2\pi}$, though the required answer is the integral part of this result. The final part was less well done, and in some cases a simple figure might have aided candidates' understanding. Two alternative ways of finding ωt are $\cos^{-1}\left(\frac{0.5}{2.5}\right)$ and one-quarter of the period plus $\sin^{-1}\left(\frac{0.5}{2.5}\right)$.

Answers: 15; 1.09 s.

Question 3

Taking moments about either *C* or *D* for the cube, and realising that the reaction of the plane on the cube acts at *D*, yields the horizontal force at *B* between the rod and cube. Taking moments about *A* for the rod relates this force to *P*, producing the given value for the latter. A significant number of candidates produced moment equations about various points with either too few or too many terms, for example moments about *A* for the system but including the force at *B*. An inequality for the coefficient of friction follows from the usual condition $F < \mu R$ where here *F* must equal the force at *B* and *R* is the weight of the cube.

Answers: $\frac{500}{3}$ N; $\mu > \frac{5}{6}$.

Question 4

The first part of the question concerns the initial motion, for which an inequality results for u from realising that the upward centripetal force must be less than the weight in order that the initial reaction is non-zero. A number of candidates, though, overlooked the word 'immediately' in the question and considered the reaction only at a general point during the ensuing motion. This is of course helpful in the second part of the question, where it is also necessary to use conservation of energy to relate the speed at a general point to u. Combining this with the reaction shows that the latter must become zero before the line OP has turned through a right angle, though the argument can be framed in various ways.

Answer: $u < \sqrt{(ga)}$.

Question 5

Almost all candidates realised that B moves in the opposite direction with speed $\frac{1}{2}u$ after striking the barrier, and most knew in principle how to find the speeds of A and B after their mutual collision by using conservation of momentum and Newton's restitution equation. However confusion over signs and algebraic errors prevented many candidates from solving these two equations correctly for the new velocities of A and B. Inequalities for k result from requiring that each of these velocities be reversed in the collision. The magnitude of the impulse is readily found from the change in momentum of either of the spheres.

Answers: $\frac{(4-5k)u}{4(1+k)}$, $\frac{(7-2k)u}{4(1+k)}$; $k > \frac{4}{5}$, $k < \frac{7}{2}$; $\frac{9kmu}{4(1+k)}$.

Question 6

The method for calculating an unbiased estimate of the population variance and hence the required confidence interval was widely understood, and most candidates used the critical value 1.645 from the normal distribution. Those who chose to use instead a t-value should ideally have interpolated in the table since the appropriate number of degrees of freedom is here 49. It was rare to see a completely correct interpretation of 'a 90% confidence interval', namely that if the procedure were applied to a large number of similar samples then about 90% of the intervals would enclose $\mu_A - \mu_E$. More candidates, however, correctly stated in the final part that the assumption of normality is unnecessary since the sample size is large.

Answer: (–0.265, 0.749) or (–0.275, 0.759).

Question 7

This question was generally well answered. The specified tabular value results from 80(F(4) – F(3)), and most candidates realised that two columns must be combined since this expected frequency is less than 5. Depending on which two columns are combined, the calculated value of χ^2 is either 6.12 or 6.22, and comparison with the critical tabular value 4.605 leads to the conclusion that the distribution does not fit.

Question 8

Most candidates performed well on this question. The product moment correlation coefficient follows from the standard formula, and either the regression line of y on x or of x on y can be used to estimate x when $y = 100$. The nearness of the correlation coefficient to 1 caused most candidates to correctly comment that their answer is reliable, though other valid comments are possible.

Answers: **(i)** 0.945; **(ii)** 14.8 cm.

Question 9

The distribution was frequently identified as being Poisson with parameter $2.4t$, which enabled these candidates to calculate $1 - \mathrm{P}(N = 0)$ correctly. Although the required probability density function of T is then found by differentiating this expression, many candidates did not explicitly relate $\mathrm{F}(t)$ to the probability of at least one log-on occurring during t minutes. The distribution of T was often identified correctly as a negative exponential, and the mean and variance stated.

Answers: $1 - \mathrm{e}^{-2.4t}$; $2.4\,\mathrm{e}^{-2.4t}$; $\frac{5}{12}$, $\frac{25}{144}$.

Question 10

The required further assumption is that the population, and not the sample as some candidates wrongly stated, is normally distributed. Many candidates stated the null and alternative hypotheses correctly, and went on to compare the calculated value 1.84 of t with the critical tabular value 1.796, concluding that the mean IQ does indeed exceed 128.

Question 11 EITHER

This alternative was unpopular, and few complete attempts were seen despite it being relatively straightforward. In order to derive the given equation of motion, it is necessary to obtain the tension T in the string using Hooke's Law. The component of T normal to AB is then $T\cos\left(\frac{1}{2}\theta\right)$ and the particle of mass m is also subject to a force $mg\sin\theta$ in the opposite direction. The value of k follows from equating the angular acceleration to zero when $\theta = \frac{1}{3}\pi$. Although candidates frequently substituted $\frac{1}{3}\pi + \varepsilon$ into the given equation in place of θ, few were able to rearrange it appropriately in order to approximate $\cos\left(\frac{1}{2}\varepsilon\right)$ and $\cos\varepsilon$ by unity, and $\sin\left(\frac{1}{2}\varepsilon\right)$ and $\sin\varepsilon$ by $\frac{1}{2}\varepsilon$ and ε respectively so as to obtain an SHM equation.

Answers: $k = 1$; $2\pi\sqrt{\frac{8a}{3g}}$.

Question 11 OR

In this alternative all the observed values are dependent on the parameter n, while the expected values are constants, so that χ^2 is here a constant multiple of $(n - 15)^2$. Most candidates attempting this question used the correct critical value 3.841, but a significant number took χ^2 to be smaller than this tabular value rather than larger. Using the latter inequality produces the two limiting values of n, which must of course be interpreted as acceptable integers. The key to the final part is to calculate the four expected values, since one of them is found to be smaller than 5 which creates the problem of zero degrees of freedom were cells to be combined.

Answer: 1 ∅ n ∅ 11, 19 ∅ n ∅ 25.