UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Subsidiary Level and GCE Advanced Level

MARK SCHEME for the October/November 2008 question paper

9231 FURTHER MATHEMATICS

9231/01 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- CIE will not enter into discussions or correspondence in connection with these mark schemes.

CIE is publishing the mark schemes for the October/November 2008 question papers for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level syllabuses and some Ordinary Level syllabuses.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

1 $\dot{x} = 4t^3 - 4/t,\ \dot{y} = 8t$ B1

$s = \int_0^4 \left[\left(4t^3 - 4/t\right)^2 + 64t^2\right]^{1/2} dt$ M1A1

$\left[\left(4t^3 - 4/t\right)^2 + 64t^2\right]^{1/2} = 4t^3 + 4/t$ B1

$s = [t^4 + 4\ln t]_2^4 = 240 + 4\ln 2$ A1

2 MV (y wrt x) over [0, 2] = (1/2) $\int_0^2 e^x dx$ = (1/2) $[e^x]_0^2$ = $(e^2 - 1)/2$ (=3.19) M1A1

$\dfrac{\int_1^{e^2} \ln y\, dy}{e^2 - 1}$ M1

$= \dfrac{[y \ln y - y]_1^{e^2}}{e^2 - 1}$ M1A1 (for integration of ln y) – can be earned independently

$= \left[\dfrac{2e^2 - e^2}{e^2 - 1}\right] - \left[\dfrac{-1}{e^2 - 1}\right]$ (oew)

$= \dfrac{e^2 + 1}{e^2 - 1}$ A1 (AG)

3 Approximately correct curve passing through the pole, O, and the point $A(\pi^2/4,0)$. B1

Negative gradient at A B1

Correct form at O. B1

Area $= (1/2)\int_0^{\pi/2} (\pi/2-\theta)^4 \, d\theta$ M1

$= -(1/10)\left[(\pi/2-\theta)^5\right]_0^{\pi/2}$ A1

$= \pi^5/320$ A1

4 $\mathbf{Ae} = \lambda\mathbf{e}$ B1

$\mathbf{A}^2\mathbf{e} = \mathbf{A}(\mathbf{A})\mathbf{e} = \mathbf{A}(\lambda\mathbf{e}) = \lambda(\mathbf{Ae}) = \lambda^2\mathbf{e} \Rightarrow$ eigenvalue is λ^2 M1A1

$\mathbf{Ae} = 3\mathbf{e}$ for some $\mathbf{e}$

$\Rightarrow (\mathbf{A}^4 + 3\mathbf{A}^2 + 2\mathbf{I})\mathbf{e} = 81\mathbf{e} + 27\mathbf{e} + 2\mathbf{e} = 110\mathbf{e}$ M1M1

$\Rightarrow$ an eigenvalue is 110 A1

OR

3 is an eigenvalue of $\mathbf{A}$

$\therefore\ 3^2 = 9$ is an eigenvalue of $\mathbf{A}^2$
and $3^4 = 81$ is an eigenvalue of $\mathbf{A}^4$ (either of these) M1

eigenvalue of $\mathbf{A}^4 + 3\mathbf{A}^2 + 2\mathbf{I}$
$= 81 + 3\times 9 + 2$ (Adding $\geqslant$2 terms) M1

$= 110$ A1

5 $2x - xy_1 - y - 4yy_1 = 0$ B1

$\Rightarrow \ldots y_1(2) = 2$ (AG) B1

$2 - xy_2 - y_1 - y_1 - 4y_1^2 - 4yy_2 = 0$ M1A2

$2 - 2y_2(2) - 2 - 2 - 16 = 0 \Rightarrow y_2(2) = -9$ M1A1

OR

$$2x - y = (4y + x)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x - y}{4y + x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(4x + y)(2 - y_1) - (2x - y)(1 + 4y_1)}{(x + 4y)^2}$$

M1 Use of quotient rule
A1 ⩾ term of numerator
A1 All correct

$$= \frac{8 \times 0 - 4 \times 9}{4}$$

M1 Substitution of values

$= -9$ A1

6 **(i)** Reduction of **A** to echelon form, e.g.,

$$\begin{pmatrix} 1 & -1 & -2 & -3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & \alpha - 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

M1A1

$\alpha = 9 \Rightarrow$ last 2 rows consist entirely of zeros $\Rightarrow r(\mathbf{A}) = 2$ A1

A basis for the null space of **A** is $\left\{ \begin{pmatrix} 5 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} \right\}$, or equivalent M1A1

(ii) $\alpha - 9 \neq 0$ M1

$r(\mathbf{A}) = 3$ A1

7 $D[x(1+x^4)^{-n}] = (1+x^4)^{-n} - 4nx^4(1+x^4)^{-n-1}$ M1

$= (1-4n)(1+x^4)^{-n} + 4n(1+x^4)^{-n-1}$ A1

$\Rightarrow \left[\dot{x}(1+x^4)^{-n}\right]_0^1 = (1-4n)I_n + 4nI_{n+1}$ M1

$\Rightarrow 4nI_{n+1} = 2^{-n} + (4n-1)I_n$ (AG) A1

$8I_3 = 1/4 + 7I_2,\ 4I_2 = 1/2 + 3I_1$ B1B1

$I_3 = 9/64 + (21/32)I_1 \approx 0.7096$ or 0.710 M1A1

OR

$n = 1 \quad 4I_2 = \dfrac{1}{2} + 3 \times 0.86697 \Rightarrow I_2 = 0.7752275$

$n = 2 \quad 8I_3 = \dfrac{1}{4} + 7 \times 0.7752275 \Rightarrow I_3 = 0.7095740625$

$\therefore I_3 = 0.7096$ or 0.710

(No penalty for correct 5 dp value.)

M1	Use of formula
A1	Gets I_2
A1ft	Subs value for I_2 in I_3 formula
A1	obtains I_3 correct (cao)

8 AQE has roots $-3/5 \pm (4/5)\text{i}$ M1

CF: $e^{-3t/5}\,[A\cos(4t/5) + B\sin(4t/5)]$ A1

PI = $at^2 + bt + c \Rightarrow 10a + 6(2at + b) + 5(at^2 + bt + c) \equiv 5t^2 + 12t + 15$ M1

$5a = 5,\ 12a + 5b = 12,\ 10a + 6b + 5c = 15$ A1

$\Rightarrow a = 1,\ b = 0,\ c = 1$ A1

GS: $y = e^{-3t/5}\,[A\cos(4t/5) + B\sin(4t/5)] + t^2 + 1$ A1

$y(0) = 0 \Rightarrow 0 = A + 1 \Rightarrow A = -1$ B1

$\dot{y} = -(3/5\ e^{-3t/5}\,[A\cos(4t/5) + B\sin(4t/5)] + e^{-3t/5}\,[(-4A/5)\sin(4t/5) + (4B/5)\cos(4t/5)] + 2t$

$\Rightarrow \dot{y}(0) = 0 \Rightarrow -3A/5 + 4B/5 = 0$ M1

$\Rightarrow B = -3/4 \Rightarrow y = -(1/4)e^{-3t/5}\,[(4\cos(4t/5) + 3\sin(4t/5)] + t^2 + 1$ A1

or $-1.25\cos(0.8t - 0.64)$

or $1.25\cos(0.8t + 2.50)$ etc.

9 Set up

$$H_k : \sum_{n=1}^{k} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)}$$ B1

for some positive integer k

$$H_k \Rightarrow \sum_{n=1}^{k+1} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)} + \frac{4k+5}{(k+1)(k+2)(2k+1)(2k+3)}$$ M1

$$= 1 - \frac{2k^2+3k+1}{(k+1)(k+2)(2k+1)(2k+3)}$$ A1

$$= ... = 1 - \frac{1}{(k+2)(2k+3)}$$ A1

Verifies H_1 is true. B1

Correct completion of induction argument A1

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = ... = \frac{1}{(N+1)(2N+1)} - \frac{1}{(2N+1)(4N+1)}$$ M1A1

$$= \frac{3N}{(N+1)(2N+1)(4N+1)} < \frac{3N}{N.2N.4N} = \frac{3}{8N^2}$$ M1A1

OR

$$= \frac{3N}{8N^3+14N^2+7N+1} = \frac{3}{8N^2+14N+7+\frac{1}{N}}$$

Since $N \geqslant 1$ $\;14N+7+\frac{1}{N} > 0$

$$\therefore \sum < \frac{3}{8N^2}$$

10 Write $c = \cos\theta,\ s = \sin\theta,\ c_n = \cos(n\theta)$

$c_8 + \mathrm{i}s_8 = (c + is)^8 \Rightarrow c_8 = c^8 - 28c^6s^2 + 70c^4s^4 - 28c^2s^6 + s^8$ M1A1

$\Rightarrow c_8 = c^8 - 28c^6(1-c^2) + 70c^4(1-2c^2+c^4) - 28c^2(1-3c^2+3c^4-c^6) + (1-4c^2+6c^4-4c^6+c^8)$
M1A1

$\Rightarrow c_8 = 128c^8 - 256c^6 + 160c^4 - 32c^2 + 1$ (*) A1

(i) $\theta \to \pi/2 - \theta$ in (*) leads to:

$c_8 = 128s^8 - 256s^6 + 160s^4 - 32s^2 + 1$ M1A1

(ii) From (*), $x = \cos^2 \pi/8 \Rightarrow 32(4x^4 - 8x^3 + 5x^2 - x) + 1 = \cos\pi = -1$ M1M1

$\Rightarrow 4x^4 - 8x^3 + 5x^2 - x = -1/16$ A1

11 (2**j** – **k**) × (3**i** + 2**j** – 2**k**) = –2**i** – 3**j** – 6**j** (oew) M1A1

$\Pi_1 : 2x + 3y + 6z = 14$ (AG) M1A1

Perpendicular distance, *p*, of *P* from l in terms of 1 parameter, e.g.,

$p = (1/7)\left|2(3+4\lambda) + 3(8+6\lambda) + 6(2+5\lambda) - 14\right|$ M1

$= \left|4 + 8\lambda\right|$ A1

$p \leqslant 4 \Rightarrow -1 \leqslant \lambda \leqslant 0$ M1A1

(3**i** + 8**j** + 2**k**) – (**i** + 2**j** + **k**) = 2**i** + 6**j** + **k**

(2**i** + 6**j** + **k**) × (4**i** + 6**j** + 5**k**) = 24**i** – 6**j** – 12**k** M1A1

$\cos\alpha = \left|(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}).(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})/7\sqrt{21}\right| = 1/\sqrt{21}$ M1

$\alpha = 77.4°$ A1

12 EITHER

(i)	$x = 1$, $x = 3$ (both)	B1
	$y = 1$	B1
(ii)	Solves $(x-2)(x-a)/(x-1)(x-3) = 1$ to obtain $x = \xi$ where $\xi = (2a-3)/(a-2)$	M1A1
(iii)	$y_1 = 0 \Rightarrow (x-2)(x-a)(2x-4) = (x-1)(x-3)(2x-2-a)$	M1
	$\Rightarrow (-4-4-2a)x^2 + (4a+8+4a)x - 8a = (-8-2-a)x^2 + (6+8+4a)x - 6 - 3a$	A1
	$\Rightarrow (a-2)x^2 + (6-4a)x + (5a-6) = 0$ (AG)	A1
	$(6-4a)^2 \geqslant 4(a-2)(5a-6)$	M1
	$\Rightarrow a^2 - 4a + 3 \leqslant 0 \Rightarrow (a-1)(a-3) \leqslant 0$	M1
	$\Rightarrow 1 < a < 3$ ($a \neq 2$ given)	A1
(iv)	**(a)** Axes and asymptotes	B1
	Branches (all)	B1
	(b) Middle branch with maximum value in the range $0 < y < 1$	B1
	Outside branches with correctly placed minimum point	B1

OR

(i)	α a root of given equation $\Rightarrow \alpha^4 - 5\alpha^2 + 2\alpha - 1 = 0$	
	$\Rightarrow \alpha^{n+4} - 5\alpha^{n+2} + 2\alpha^{n+1} - \alpha^n = 0$	M1
	Summing over α, β, γ, δ, leads to $S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0$	A1
(ii)	$S_2 = 10$	B1
	$S_4 = 5S_2 - 2S_1 + 4 = 50 - 0 + 4 = 54$	M1A1
(iii)	$S_{-1} = 2$ from e.g., $y^4 - 2y^3 + 5y^2 - 1 = 0$	M1A1
	$S_3 = 5S_1 - 2S_0 + S_{-1} = -6$	M1A1
	OR	
	$2S_3 = 3S_1S_2 - S_1^3 + 6\sum\alpha\beta\gamma$	M1A1
	$= 3 \times 10 \times 0 - 0 + 6 \times (-2)$	M1
	$\Rightarrow S_3 = -6$	A1
	$S_6 = 5S_4 - 2S_3 + S_2 = 292$	M1A1
(iv)	$\sum\alpha^2\beta^4 = S_2S_4 - S_6 = 540 - 292 = 248$	M2A1