CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2014 series

9231 FURTHER MATHEMATICS

9231/23 Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

CAMBRIDGE
International Examinations

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a “fortuitous” answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question Number	Mark Scheme Details		Part Mark	Total
1 (i)	Use conservation of momentum, e.g.	$mv_A + kmv_B = mu + ⅔kmu$		
		or $v_A + kv_B = u(1 + ⅔k)$ B1		
	Use restitution (4/5 on wrong side is M0;	$v_A - v_B = -(4/5)(u - ⅔u)$		
	signs inconsistent with prev. eqn is A0):	*or* $v_A - v_B = -4u/15$ M1 A1		
	Solve for v_A (allow verification):	$(1 + k)v_A = u(1 + ⅔k - 4k/15)$		
		$v_A = u(2k + 5)/5(k + 1)$ **A.G.** M1 A1		
		$[v_B = u(10k + 19)/15(k + 1)]$	5	
(ii)	Equate impulse to momentum change for *A*:	$mu - (2/5)mu = mv_A$		
		$3/5 = (2k + 5)/5(k + 1)$, $k = 2$ M1 A1		
	OR B:	$⅔kmu + (2/5)mu = kmv_B$		
		$⅔k + (2/5) = k(10k + 19)/15(k + 1)$		
		$10k^2 + 16k + 6 = 10k^2 + 19k$, $k = 2$ (M1 A1)	2	**7**

2	Find radial acceleration from $r\,(\mathrm{d}\theta/\mathrm{d}t)^2$ $[\equiv r\omega^2]$: $\mathrm{d}\theta/\mathrm{d}t = (2\sin 2t)(2\cos 2t)$ $= 2\sin 4t$ $r\,(\mathrm{d}\theta/\mathrm{d}t)^2 = 4a\sin^2 4t$ **A.G.**	M1 A1	2	
(i)	Find t by equating $\mathrm{d}^2\theta/\mathrm{d}t^2$ to 0: $\mathrm{d}^2\theta/\mathrm{d}t^2 = 8\cos 4t = 0$ $t = \pi/8$ *or* 0·393	M1 A1	2	
(ii)	Find radial and transverse components of force: $4ma\sin^2 4\pi/12 = 3ma$ *and* $8ma\cos 4\pi/12 = 4ma$ Combine to find magnitude of resultant force: $\sqrt{(3^2 + 4^2)}\,ma = 5ma$	M1 A1	2	**6**

3	(i)	Use conservation of energy:	$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mg(a - a\cos\theta)$ B1		
		Use $F = ma$ radially:	$T - mg\cos\theta = mv^2/a$ B1		
		Relate u to impulse J:	$mu = J$ B1		
		Eliminate u and v to find T:	$T = J^2/ma - mg(2 - 3\cos\theta)$ **A.G.** M1 A1	5	
	(ii)(a)	Investigate v [and T] for $k = 1$ and describe motion:	$v^2 = ga(2\cos\theta - 1)$ $[T = mg(3\cos\theta - 1)]$ $v = 0$ [and $T > 0$] when $\cos\theta = \frac{1}{2}$		
		(**S.R.** Award B1 for correct result based only on T)	so P oscillates (A.E.F.) M1 A1	2	
	(b)	Investigate v <u>and</u> T for $k = 6$ and describe motion:	$T = mg(3\cos\theta + 4)$ $[v^2 = ga(2\cos\theta + 4)]$ $T > 0$ for e.g. $\theta = \pi$ *and* $v > 0$		
		(**S.R.** Award B1 for correct result based only on T)	so P does full circle(A.E.F.) M1 A1	2	**9**

4	*EITHER:* Resolve vertically: $R_A + R_C \cos\alpha = 3mg/2$ B1 Take moments about A: $R_C\, 2a = mgd \cos\alpha + \tfrac{1}{2}mg2d \cos\alpha$ M1 A1 Eliminate R_C to find R_A: $R_C = (mgd/a)\cos\alpha = 3mgd/5a$ $R_A = 3mg/2 - 9mgd/25a$ *or* $3mg(25a - 6d)/50a$ (A.E.F.) M1 A1 *OR:* Resolve along AB: $R_A \sin\alpha + F_A \cos\alpha =$ $mg \sin\alpha + \tfrac{1}{2}mg \sin\alpha$ (B1) Take moments about C: $(R_A \cos\alpha)\, 2a - (F_A \sin\alpha)2a =$ $(mg \cos\alpha)(2a - d)$ $- (\tfrac{1}{2}mg \cos\alpha)(2d - 2a)$ (M1 A1) Eliminate F_A to find R_A: $R_A = 3mg(25a - 6d)/50a$ (A.E.F.) (M1 A1) Find limit on d from $R_A > 0$: $25a - 6d > 0,\ d < 25a/6$ **A.G.** B1			
			6	
	Find F_A by e.g. horizontal resolution: $F_A = R_C \sin\alpha = (3mgd/5a)(4/5)$ $= 12mgd/25a$ M1 A1 Find inequality for μ from $F_A \leqslant \mu R_A$ (= loses A1): $\mu \geqslant 8d/(25a - 6d)$ **A.G.** M1 A1		4	**10**

5	Find MI of rectangular lamina about *O*: $I_{\square,O} = \frac{1}{3}M\{(4a)^2 + (3a)^2\}$ B1 [$= 25\,Ma^2/3$] Find MI of circular lamina about *O*: $I_{O,O} = \frac{1}{2}\,\frac{1}{3}M\,(5a/2)^2$ B1 [$= 25\,Ma^2/24$] *EITHER*: Find MI of combined laminas about *O*: $I_O = I_{\square,A} + I_{O,A} = 225\,Ma^2/24$ M1 A1 Find MI of combined laminas about *A*: $I_A = I_O + (4M/3)\,25a^2$ $= 1025\,Ma^2/24$ M1 A1 *OR*: Find MI of rectangle about *A* $I_{\square,A} = I_{\square,O} + 25\,Ma^2 = 100\,Ma^2/3$ and of circle about *A*: *and* $I_{O,A} = I_{O,O} + \frac{1}{3}M\,25a^2 = 75\,Ma^2/8$ (M1 A1) Find MI of combined laminas about *A*: $I_A = I_{\square,A} + I_{O,A} = 1025\,Ma^2/24$ (M1 A1) Find MI of system about *A*: **A.G.** $I = I_A + 50\,Ma^2 = 2225\,Ma^2/24$ M1 A1	8	
	State or imply that speed is max when *AC* vertical M1 Use energy when *AC* vertical (or at general point): $\frac{1}{2}I\omega^2 = 4Mg/3 \times 5a + \frac{1}{2}Mg \times 10a$ *or* $11Mg/6 \times 70a/11$ [$=35Mga/3$] M1 A1 Substitute for *I* and find *k* in max ang. speed ω: $\omega^2 = (112/445)g/a,\ \ k = 0{\cdot}502$ A1	4	**12**

6	Consider differences e.g. before – after: $8\ 3\ -4\ 2\ -11\ 3\ 17\ 10\ 6\ 0$ M1 Calculate sample mean $\bar{d} = 34/10 = 3{\cdot}4$ *and* and estimate population variance: $s^2 = (648 - 34^2/10)/9$ (allow biased here: $53{\cdot}24$ *or* $7{\cdot}297^2$) $= 2662/45$ *or* $59{\cdot}16$ *or* $7{\cdot}691^2$ M1 State hypotheses (A.E.F.; B0 for $\bar{x}$), e.g.: H_0: $\mu_B - \mu_A = 0$, H_1: $\mu_B - \mu_A > 0$ B1 Calculate value of t: $t = \bar{d}/(s/\sqrt{10}) = 1{\cdot}398$ *or* $1{\cdot}4$ M1 A1 State or use correct tabular t-value: $t_{9,\,0.9} = 1{\cdot}38[3]$ B1 (or can compare $\bar{d}$ with 3·36[4]) Consistent conclusion (A.E.F, √ on both t–values): [Reject H_0:] Hours of absence have decreased B1√ Wrong test can earn only B1 for hypotheses and B1 for conclusion	7	**7**

7	**(i)**	Find $P(N = 5)$: $P(N = 5) = (3/4)^4 \times 1/4$ $= 81/1024$ *or* $0{\cdot}079[1]$ B1	1	
	(ii)	Find $P(N > 8)$: $P(N > 8) = (3/4)^8$ $= 6561/65536$ *or* $0{\cdot}1[00]$ M1 A1	2	
		Find prob. P_J that James qualifies: $P_J = P(N \leqslant 6) = 1 - (3/4)^6$ *or* $\{1 + 3/4 + (3/4)^2 + (3/4)^3 + (3/4)^4 + (3/4)^5\}1/4$ $= 3367/4096$ *or* $0{\cdot}822$ M1 A1	2	
		Find prob. P_C that Colin qualifies: $P_C = 1 - (2/3)^6$ [$= 0{\cdot}9122$] B1 Find prob. that exactly one qualifies: $P_J(1 - P_C) + P_C(1 - P_J)$ $= (3367/4096)(64/729)$ $+ (665/729)(729/4096)$ $= 0{\cdot}0722 + 0{\cdot}1624$ $= 0{\cdot}235$ (allow $0{\cdot}234$) M1 A1	3	**8**

8	Find expected values (to 1 d.p.): 32·4 15·3 12·3 40·5 19·125 15·375 35·1 16·575 13·325 M1 A1 State (at least) null hypothesis (A.E.F.): H_0: Car type is independent B1 Calculate χ^2 (to 1 d.p.): χ^2 = 0·005 + 1·208 + 1·796 + 0·500 + 1·243 + 5·716 + 0·479 + 0·020 + 1·640 = 12·6 (allow 12·7) M1 A1 State or use correct tabular χ^2 value (to 3 s.f.): $\chi_{4,\,0.95}{}^2$ = 9·488 *OR* 9·49 B1 Valid method for reaching conclusion: Reject H_0 if χ^2 > tabular value M1 Conclusion consistent with correct values (A.E.F): Car type is dependent on age A1	8	**8**

9	Find k for which $P(X \geqslant k) = 0{\cdot}6$: $\quad 0{\cdot}6 = 1 - F(k)$ M1			
	$= 1 - (k/8 - ¼)$ M1			
	$k = 26/5$ *or* $5{\cdot}2$ A1		3	
	Find $G(y)$ from $Y = 2 \ln X$ for $2 \leqslant x \leqslant 10$: $\quad G(y) = P(Y < y) = P(2 \ln X < y)$			
	(allow < or ⩽ throughout) $\quad = P(X < e^{y/2}) = F(e^{y/2})$			
	(result may be stated) $\quad = e^{y/2}/8 - ¼ \; (2 \ln 2 \leqslant y \leqslant 2 \ln 10)$			
	or $(\ln 4 \leqslant y \leqslant \ln 100)$			
	or $(1{\cdot}39 \leqslant y \leqslant 4{\cdot}61)$ M1 A1			
	State $G(y)$ for other values of x: $\quad 0 \; (y < 2 \ln 2)$ *and* $1 \; (y > 2 \ln 10)$ B1		3	
	Find $g(y)$ for $2 \ln 2 \leqslant y \leqslant 2 \ln 10$: $\quad g(y) = e^{y/2}/16$ M1 A1			
	Sketch positive exponential for $2 \ln 2 \leqslant y \leqslant 2 \ln 10$ B1			
	Show $g(y) = 0$ on either side of this interval B1		4	**10**

10 **(i)**	Calculate sample mean:	$\bar{x} = 132/8 = 16{\cdot}5$ M1		
	Estimate population variance:	$s^2 = (2192{\cdot}06 - 132^2/8)/7$		
	(allow biased here: $1{\cdot}757$ *or* $1{\cdot}326^2$)	$= 703/350$ *or* $2{\cdot}009$ *or* $1{\cdot}417^2$ M1		
	State hypotheses (A.E.F.; B0 for $\bar{x}$):	H_0: $\mu = 15{\cdot}8$, H_1: $\mu > 15{\cdot}8$ B1		
	Calculate value of t (to 3 s.f.):	$t = (\bar{x} - 15{\cdot}8)/(s/\sqrt{8}) = 1{\cdot}4[0]$ M1 A1		
	State or use correct tabular t–value (to 3 s.f.):	$t_{7,\,0.9} = 1{\cdot}41[5]$ B1		
	(or can compare $\bar{x}$ with $15{\cdot}8 + 0{\cdot}709 = 16{\cdot}51$)			
	Consistent conclusion (A.E.F, √ on both t–values):	[Accept H_0:] Popn. mean not greater than $15{\cdot}8$ B1√	7	
(ii)	Find confidence interval (allow z in place of t) e.g.:	$16{\cdot}5 \pm t\sqrt{(2{\cdot}009/8)}$ M1 A1		
	(Use of 15·8 does not lose M1)			
	Use of correct tabular value:	$t_{7,\,0.975} = 2{\cdot}36[5]$ A1		
	Evaluate C.I. correct to 3 s.f.:	$16{\cdot}5 \pm 1{\cdot}18[5]$ *or* [15·3, 17·7 A1	4	**11**

11 A (i)	*EITHER:* Find extension e at equilibrium point: $e = 5l/7$ B1 Resolve forces at eq. pt. to find modulus λ: $\lambda e/l = mg, \; \lambda = 7mg/5$ **A.G.** B1 *OR:* Use conservation of energy to find λ: $\frac{1}{2}\lambda\{(6l/7)^2 - (4l/7)^2\}/l = mg2l/7$ $\lambda = 7mg/5$ **A.G.** (M1 A1)	2	
(ii)	Use Newton's Law at general point: $m\,\mathrm{d}^2x/\mathrm{d}t^2 = mg - \lambda(e+x)/l$ [*or* $-mg + \lambda(e-x)/l$] M1 A1 Simplify to give standard SHM eqn: $\mathrm{d}^2x/\mathrm{d}t^2 = -\lambda x/lm$ *or* $-7gx/5l$ A1 S.R.: Stating this without derivation (max 2/4): (B1) Find period $T = 2\pi/\omega$ with $\omega^2 = 7g/5l$: $T = 2\pi/\sqrt{(7g/5l)}$ *or* $2\pi\sqrt{(5l/7g)}$ B1	4	

(iii)	*EITHER*: Equate *P*'s speed to one-half its max. speed: $\omega^2(l^2/7^2 - x^2) = \frac{1}{4}\omega^2 l^2/7^2$, M1 A1 Find *x*: (A.E.F.) $x^2 = \frac{3}{4} l^2/7^2$, $x = (\sqrt{3}/14)\, l$ A1 *either:* Find *t* from $x = x_0 \cos \omega t$: $t = \omega^{-1} \cos^{-1}\{(\sqrt{3}/14)l/(l/7)\}$ M1 A1 $= \omega^{-1}\cos^{-1}(\sqrt{3}/2) = \pi/6\omega$ $= (\pi/6)\sqrt{(5l/7g)}$ A1 *or:* Find *t* from $x = x_0 \sin \omega t$: $t = \frac{1}{4}T - \omega^{-1}\sin^{-1}\{(\sqrt{3}/14)l/(l/7)\}$ (M1 A1) $= \frac{1}{4}T - \omega^{-1}\sin^{-1}(\sqrt{3}/2)$ $= (\pi/2 - \pi/3)\sqrt{(5l/7g)}$ $= (\pi/6)\sqrt{(5l/7g)}$ (A1) *OR:* Equate *P*'s speed to one-half its max. speed: $\omega x_0 \sin \omega t$ *or* $\omega x_0 \cos \omega t = \frac{1}{2}\omega x_0$ (M1 A1; A1) Find first value of *t*: $t = \omega^{-1}\sin^{-1}(1/2)$ *or* $\frac{1}{4}T - \omega^{-1}\cos^{-1}(1/2)$ (M1 A1) $= (\pi/6)\sqrt{(5l/7g)}$ (A1)	6	**12**

11 (b) (i)	State both hypotheses (B0 for r …): H_0: $\rho = 0$, H_1: $\rho > 0$ B1 State or use correct tabular one-tail r-value: $r_{12,\,5\%} = 0{\cdot}497$ *B1 Valid method for reaching conclusion: Reject H_0 if $0{\cdot}6 >$ tabular value M1 Consistent conclusion (A.E.F, √ on 0·497): There is positive correlation A1√	4	
(ii)	Use $r^2 = ab$ to eliminate a or b: $0{\cdot}5 = 0{\cdot}6^2/a - a$ or $b - 0{\cdot}6^2/b$ M1 Solve and select correct values: $a^2 + 0{\cdot}5a - 0{\cdot}36 = 0$ or $b^2 - 0{\cdot}5b - 0{\cdot}36 = 0$ $(a + 0{\cdot}9)(a - 0{\cdot}4) = 0$ or $(b + 0{\cdot}4)(b - 0{\cdot}9) = 0$ (A0 if $a = -0{\cdot}9$, $b = -0{\cdot}4$ not rejected) $a = 0{\cdot}4$ and $b = 0{\cdot}9$ M1 A1	3	
(iii)	Use $\bar{x}$ [= 5·5] in 1st regression eqn to find $\bar{y}$: $\bar{y} = (66/12)\,b + 4{\cdot}5$ [= 9·45] M1 Use $\bar{x}$ and $\bar{y}$ in 2nd regression eqn to find c: $c = \bar{x} - a\,\bar{y} = 1{\cdot}72$ B1 Sketch both regression lines on one diagram $y = 0{\cdot}9x + 4{\cdot}5$ and $x = 0{\cdot}4y + 1{\cdot}72 \Rightarrow y = 2{\cdot}5x - 4{\cdot}3$ B1	3	
(iv)	State coefficient of x in eqn of z on x (√ on b): $5 \times b = 4{\cdot}5$ B1√ State value of r with valid justification, e.g.: $r = \sqrt{(4{\cdot}5 \times (0{\cdot}4/5))} = 0{\cdot}6$ or r is unchanged by scaling so 0·6 B1	2	12