Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/23**

Paper 2 **May/June 2016**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a “fortuitous” answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

<u>Penalties</u>

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question Number	Mark Scheme Details	Part Mark	Total
1	Relate u^2, v^2 at lowest and highest points by energy: $\frac{1}{2} mv^2 = \frac{1}{2} mu^2 - 2mga$ $[\ v^2 = u^2 - 4ag\ ]$	B1	
	Find tension T_1 at lowest point by using $F = ma$: $T_1 = mu^2/a + mg$	B1	
	Find tension T_2 at highest point by using $F = ma$: $T_2 = mv^2/a - mg$	B1	
	EITHER: Relate u^2, v^2 using $T_1 = 2T_2$: $2v^2 = u^2 + 3ag$	M1	
	Eliminate v^2 to find u: $u = \sqrt{(11ag)}$	M1 A1	[6]
	OR: Relate T_2, u^2 by substituting for v^2: $T_2 = mu^2/a - 5mg$	(M1)	
	Use $T_1 = 2T_2$ to find u:$u = \sqrt{(11ag)}$ $[v = \sqrt{(7ag)}]$	(M1 A1)	
2	Take moments for rod about e.g. A: $R_P \times AP = W \times a \cos\theta + 2W \times 2a \cos\theta$ $[= 5Wa \cos\theta = 3Wa\ ]$ or P: $R_A \times AP \cos\theta - F_A \times AP \sin\theta = W \times (AP - a) \cos\theta - 2W \times (2a - AP) \cos\theta$ or B: $R_A \times 2a \cos\theta - F_A \times 2a \sin\theta + R_P \times (2a - AP) = W \times a \cos\theta$ or midpoint of AB: $R_P \times (AP - a) - R_A \times a \cos\theta + F_A \times a \sin\theta = 2W \times a \cos\theta$ or below B: $R_A \times 2a \cos\theta + R_P \cos\theta \times (2a–AP) \cos\theta - R_P \sin\theta \times AP \sin\theta = W \times a \cos\theta$	M1 A1	
	Formulate 2 more independent eqns. for forces, e.g.: Resolve forces on rod horizontally: $F_A = R_P \sin\theta [= 4R_P/5]$	B1	
	Resolve forces on rod vertically: $3W - R_A = R_P \cos\theta [= 3R_P/5]$ (or further independent moment eqn.)	B1	
	Eliminate θ from 3 independent eqns. for forces $[\sin\theta = 4/5,\ \cos\theta = 3/5\ ]$	M1	
	Relate F_A and R_A : $F_A = \frac{2}{3} R_A$	M1	
	Find AP from eqns. in R_P, R_A and F_A: $[R_P = 5W/3, F_A = 4W/3,\ R_A = 2W]$ $AP = 9a/5$ *or* $1{\cdot}8a$	M1 A1	[8]
3	Find c by equating transverse acceln. to 2 at $t = 3$: $2 \times 3 + c = 2, c = -4$	M1 A1	
	EITHER: Find radial force at $t = 3$: $F_R = 0{\cdot}2\ (3^2 + 3c + d)^2 / 0{\cdot}5$ $[= 0{\cdot}4\ (d - 3)^2]$	M1 A1	
	Find transverse force at $t = 3$ by $F = ma$:	B1	
	Equate $F_T^2 + F_R^2$ to $\{(2\sqrt{17})/5\}^2$: $0{\cdot}4^2 + 0{\cdot}4^2 (3^2 + 3c + d)^4 = 68/25$ $[F_R^2 = 2{\cdot}56,\ F_R = 1{\cdot}6]$	M1 A1	
	OR: Find radial acceln. at $t = 3$: $a_R = (3^2 + 3c + d)^2 / 0{\cdot}5$ $[= 2\ (d - 3)^2]$	(M1 A1)	
	Valid use below of transverse acceln. at $t = 3$: $a_T = 2$	(B1)	
	Equate $a_T^2 + a_R^2$ to $\{(2\sqrt{17})/(5 \times 0{\cdot}2)\}^2$: $2^2 + 2^2 (3^2$ $[a_R^2 = 64,\ a_R = 8]$	(M1 A1)	
	Find d by solving with $c = -4$, e.g.: $(3^2 + 3c + d)^4 = 17 - 1 = 16$ (both values reqd. for A1) $(d - 3)^2 = 4, d = 1$ *or* 5	A1	[8]
	(Omitting transverse comp. may earn M1 A1 M1 A1 B0 M0, max 4/8)		

Question Number	Mark Scheme Details		Part Mark	Total
4	Find or state MI of rod *AB* about axis at *A*:	$I_{AB} = \frac{1}{3}\, 4m\,(3a)^2 + 4m\,(3a)^2$		
		or $(4/3)\, 4m\,(3a)^2$ $[= 48\, ma^2]$	M1 A1	
	Find MI of disc about axis at *A*:	$I_{\text{disc}} = \frac{1}{2}\, m\,(2a)^2 + m\,(8a)^2$		
	(M1 for either)	$[= 66\, ma^2]$	A1	
	Find MI of object about axis at *A*:	$I = (48 + 66)\, ma^2 = 114\, ma^2$ **A.G.**	A1	[4]
	Find angular speed ω when *AB* vertical by energy:	$\frac{1}{2} I\, \omega^2$		
		$= (4 \times 3 + 8)\, mga\,(1 - \cos 60°)$		
		or $(5mg \times 4a)\,(1 - \cos 60°)$	M1 A1	
	(AEF)	$\omega^2 = 10mga / 57\, ma^2 = 10g/57a$	A1	
	Find speed of *B* from $r\omega$ (AEF)	$:6a\, \sqrt{(10g/57a)}$		
		$= \sqrt{(120ag/19)}$ *or* $2{\cdot}51\sqrt{(ag)}$		
		[allow $7{\cdot}95\sqrt{a}(g = 10)$		
		or $7{\cdot}87\sqrt{a}(g = 9{\cdot}8$ or $9{\cdot}81)$]	M1 A1	[5]
5	For *A* & *B* use conservation of momentum, e.g.:	$5mv_A + mv_B = 5mu$	M1	
	(*m* may be omitted here and below)			
	Use Newton's law of restitution (consistent signs):	$v_B - v_A = (4/5)\, u$	M1	
	Combine to find/verify v_A and find v_B, e.g.:	$6v_A = (5 - 4/5)\, u$,		
		$v_A = 7u/10$ **A.G.** *and* $v_B = 3u/2$	A1	[3]
(i)	For *B* & *C* use conservation of momentum, e.g.:	$mv_B' + kmv_C = mv_B$	M1	
	Use Newton's law of restitution (consistent signs):	$v_C - v_B' = (4/5)\, v_B$	M1	
	Combine to find *k* using $v_C = v_A$, e.g.:	$(k + 1)\, v_C = (1 + 4/5)\, v_B$		
		$k + 1 = 27/7,\ k = 20/7$ **A.G.**	M1 A1	[4]
(ii)	Find v_B' (may be found earlier):	$v_B' = -u/2$	B1	
	EITHER: Find final KE, E_{final}, of 3 particles:	$\frac{1}{2}\, 5mv_A^2 + \frac{1}{2}\, mv_B'^2 + \frac{1}{2}\, kmv_C^2$	M1	
		$= \frac{1}{2}\, mu^2\,(49/20 + 1/4 + 7/5)$		
		$= (41/20)\, mu^2$ *or* $2{\cdot}05\, mu^2$	A1	
	OR: Find 1st loss/gain in KE (M1 for either):	$L_1 = \frac{1}{2}\, 5mu^2 - \frac{1}{2}\, 5mv_A^2 - \frac{1}{2}\, mv_B^2$	(M1)	
	Find 2nd loss/gain in KE (A1 for both):	$L_2 = \frac{1}{2}\, mv_B^2 - \frac{1}{2}\, mv_B'^2 - \frac{1}{2}\, kmv_C^2$	(A1)	
		$[L_1 = (3/20)\, mu^2,\ L_2 = (3/10)\, mu^2]$		
	Find loss in KE (ignore sign for M1)	$\frac{1}{2}\, 5mu^2 - E_{final}$ *or* $L_1 + L_2$	M1	
		$[= (9/20)\, mu^2]$		
	Hence find percentage loss in KE:	$100\,(9/20)\, mu^2 / (\frac{1}{2}\, 5mu^2) = 18\%$	A1	[5]
6	State mean and variance of *N*:	$E(N) = 1/\frac{1}{4} = 4$	B1	
		$\text{Var}(N) = \frac{3}{4} / (\frac{1}{4})^2 = 12$	B1	[2]
(i)	Find prob. of succeeding in at most 4 attempts:	$P(N \le 4) = 1 - (\frac{3}{4})^4$		
		or $\frac{1}{4}\, \{1 + \frac{3}{4} + (\frac{3}{4})^2 + (\frac{3}{4})^3\}$		
		$= 175/256$ *or* $0{\cdot}684$	M1 A1	[2]
(ii)	Find prob. of succeeding in more than 6 attempts:	$P(N > 6) = (\frac{3}{4})^6$		
		$= 729/4096$ *or* $0{\cdot}178$	B1	[1]

Question Number	Mark Scheme Details		Part Mark	Total
7 (i)	Find F(t) by integration:	$F(t) = 1 - e^{-\lambda t}$	**B1**	
	Estimate λ by equating 1 – F(12) to 280/1000:	$e^{-12\lambda} = 0{\cdot}28$		
		$\lambda = (-\ln 0{\cdot}28) / 12 = 0{\cdot}106$	***M1 A1**	[3]
(ii)	Estimate mean μ of T:	$\mu = 1/\lambda = 9{\cdot}43$	**B1**	[1]
(iii)	Relate new exponential parameter λ' to λ:	$\lambda' = 1/1{\cdot}25\mu = \lambda/1{\cdot}25$ *or* $0{\cdot}8\lambda$		
	(B1 dependent on *M1)	[$= 0{\cdot}08486$ *or* $11{\cdot}78^{-1}$]	**B1**	
	Find P($t \le 12$):	$P(t \le 12) = 1 - e^{-12\lambda'}$	**M1**	
		$= 1 - e^{-1{\cdot}018}$ *or* $1 - 0{\cdot}28^{0{\cdot}8}$	**A1**	
	Find percentage:	$= 63{\cdot}9\%$	**A1**	[4]
8	State assumption, e.g.:	Normal distribution(s)	**B1**	
	State hypotheses, e.g.:	H_0: $\mu_{Before} - \mu_{After} = 0$,		
	(AEF; B0 for $\bar{x}$ … or if ambiguous)	H_1: $\mu_{Before} - \mu_{After} > 0$	**B1**	
	Consider differences e.g. Before – After:	0·7 3·0 –0·6 0·2 0·3 0·9 0 1·5	**M1**	
	Calculate sample mean	$\bar{d} = 6 / 8 = 0{\cdot}75$ *and*		
	and estimate population variance:	$s^2 = (13{\cdot}04 - 6^2/8) / 7$		
	(allow biased here: $1{\cdot}0675$ *or* $1{\cdot}033^2$)	$= 61/50$ *or* $1{\cdot}22$ *or* $1{\cdot}105^2$	**M1**	
	Calculate value of t:	$t = \bar{d}/(s/\sqrt{8}) = 1{\cdot}92$	**M1 A1**	
	State or use correct tabular t-value:	$t_{7,\,0.975} = 2{\cdot}36[5]$	**B1**	
	(or compare $\bar{d}$ with $t_{7,\,0.975}\, s/\sqrt{8} = 0{\cdot}924$)			
	Consistent conclusion(AEF, √ on both t-values):	[Accept H_0:]		
		No improvement in athletes' times	**B1**√	
	Wrong (hypothesis) test can earn only: B1 for assumption,			
	B1 for hypotheses			[8]
9	State (at least) null hypothesis (AEF):	H_0: No difference in preferences	**B1**	
	Find expected values (to 1 d.p.):	21·86718·13329·33310·667		
	(lose A1 if one error or rounded to integers)	19·13315·86725·6679·333	**M1 A2**	
	Calculate value of χ^2 (to 3 s.f.):	$\chi^2 = 0{\cdot}449 + 2{\cdot}075 + 0{\cdot}015 + 0{\cdot}510$		
		$+ 0{\cdot}513 + 2{\cdot}371 + 0{\cdot}017 + 0{\cdot}583$		
		$= 6{\cdot}53$ (allow 6·52)	**M1 A1**	
	State or use correct tabular χ^2 value (to 3 s.f.):	$\chi_{3,\,0.9}^{\;2} = 6{\cdot}25[1]$	**B1**	
	State or imply valid method for conclusion e.g.:	Reject H_0 if $\chi^2 \geqslant$ tabular value	**M1**	
	Correct conclusion, from correct values: (AEF)	Difference in preferences	**A1**	[9]

Question Number	Mark Scheme Details		Part Mark	Total
10 (i)	Find correlation coefficient r :	$r = S_{xy} / \sqrt{(S_{xx} S_{yy})}$ with e.g. $S_{xy} = 872 - 68 \times 63/5 = 15{\cdot}2$ (*or* 3·04) $S_{xx} = 942 - 68^2/5 = 17{\cdot}2$ (*or* 3·44) $S_{yy} = 815 - 63^2/5 = 21{\cdot}2$ (*or* 4·24)	**M1 A1**	
	(working may be implied)	$r = 0{\cdot}796$	***A1**	[3]
(ii)	State both hypotheses (B0 for r …):	H_0: $\rho = 0$, H_1: $\rho > 0$	**B1**	
	State or use correct tabular one-tail r-value:	$r_{5,\ 5\%} = 0{\cdot}805$	***B1**	
	State or imply valid method for conclusion e.g.:	Accept H_0 if $r <$ tab. value (AEF)	**M1**	
	Correct conclusion(AEF, dep *A1, *B1):	No positive correlation	**A1**	[4]
(iii)	Calculate reqd. values for combined sample:	$S_{xy} = 2092 - 164 \times 149/12 = 55{\cdot}67$ $S_{xx} = 2308 - 164^2/12 = 66{\cdot}67$ [$S_{yy} = 1911 - 149^2/12 = 60{\cdot}92$]		
	Calculate gradient p in $y - \bar{y} = p(x - \bar{x})$:	$p = S_{xy} / S_{xx} = 0{\cdot}835$	**M1 A1**	
	and find q:	$q = 149/12 - p \times 164/12$ $= 12{\cdot}42 - 0{\cdot}835 \times 13{\cdot}67 = 1{\cdot}00[5]$	**M1 A1**	[4]
(iv)	Estimate mark from regression line when $x = 13$:	$y = 11{\cdot}9$so mark = 12	**B1**	
	Consistent valid comment on reliability, e.g.: ($\sqrt{}$ on r') *or* *or*	Reliable since 13 (*or* 12) in range Reliable since $r' = 0{\cdot}874 \approx 1$ Reliable since $r' > r_{12,\ 5\%} = 0{\cdot}497$	**B1**$\sqrt{}$	[2]
11 (a) (i)	Find AP_0 by equating equilibrium tensions and wt.: (Lose A1 for one incorrect term) *or* if $e = AP_0 - 0{\cdot}8$:	$8(AP_0 - 0{\cdot}8)/0{\cdot}8$ $= 4(2 - AP_0)/1{\cdot}2 + 0{\cdot}2g$ $8e/0{\cdot}8 = 4(1{\cdot}2 - e)/1{\cdot}2 + 0{\cdot}2g$ [$e = 9/20$ *or* 0·45]	**M1 A2**	
		$AP_0 = 5/4$ *or* 1·25 [m]	**A1**	[4]
(ii)	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term) *or*	$\pm 0{\cdot}2\ \mathrm{d}^2x/\mathrm{d}t^2 = -8(AP_0 - 0{\cdot}8 + x)/0{\cdot}8$ $+ 4(2 - AP_0 - x)/1{\cdot}2 + 0{\cdot}2g$ $\pm 0{\cdot}2\ \mathrm{d}^2y/\mathrm{d}t^2 = 8(AP_0 - 0{\cdot}8 - y)/0{\cdot}8$ $- 4(2 - AP_0 + y)/1{\cdot}2 + 0{\cdot}2g$	**M1 A2**	
	Simplify to give standard SHM eqn, e.g.: **SC**: B1 if no derivation (max 3/6)	$\mathrm{d}^2x/\mathrm{d}t^2 = -(10x + 10x/3) / 0{\cdot}2$ $= -200x / 3$	**A1**	
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(200/3)}$:	$T = 2\pi\sqrt{(3/200)} = \pi\sqrt{(3/50)}$ *or* $0{\cdot}245\pi$ *or* 0·770 [s]	**M1 A1**	[6]
(iii)	Find or state displacement x at midpoint:	$x = 1{\cdot}6 - AP_0$[= 0·35]	**B1**	
	Find or state amplitude a:	$a = 1{\cdot}65 - AP_0$ [= 0·4]	**B1**	
	Find speed v from $v^2 = \omega^2(a^2 - x^2)$:	$v^2 = (200/3)(0{\cdot}4^2 - 0{\cdot}35^2)$ $v = \sqrt{2{\cdot}5}, v = 1{\cdot}58$ [m s^{-1}]	**M1 A1**	[4]

Question Number	Mark Scheme Details		Part Mark	Total
11 (b)	State hypotheses (B0 for $\overline{x}_A$ …), e.g.:	H_0: $\mu_B = \mu_A$,H_1: $\mu_B > \mu_A$	B1	
	Estimate both popn. variances:	$s_A^2 = (1910{\cdot}8 - 256^2/40) / 39$		
	and	$s_B^2 = (3148{\cdot}8 - 382{\cdot}9^2/50) / 49$	M1 A1	
	(allow biased here: $6{\cdot}81$ *or* $2{\cdot}610^2$	$s_A^2 = 6{\cdot}985$ *or* $454/65$ *or* $2{\cdot}643^2$		
	and $4{\cdot}331$ *or* $2{\cdot}081^2$) *and*	$s_B^2 = 4{\cdot}419$ *or* $2{\cdot}102^2$	A1	
	Estimate combined variance (may be implied):	$s^2 = s_A^2/40 + s_B^2/50$	M1	
	(to 3 s.f)	$= 0{\cdot}263$ *or* $0{\cdot}513^2$	A1	
	Calculate value of z (or $-z$):	$z = (\overline{x}_B - \overline{x}_A) / s$		
		$= (7{\cdot}658 - 6{\cdot}4) / s$		
	(to 3 s.f.)	$= 1{\cdot}258/0{\cdot}5128 = 2{\cdot}45$	M1 A1	
	State or use correct tabular z value:(to 3 s.f.)	$z_{0.99} = 2{\cdot}326$(**SC**: allow $2{\cdot}358$)	*B1	
	(or can compare $\overline{x}_B - \overline{x}_A = 1{\cdot}258$ with $1{\cdot}19$)			
	Correct conclusion (AEF, √ on z, dep *B1)	:z > tabular value		
		so μ_B is greater than μ_A	B1√	
	Note: Basing test on equal variances can earn first 4 marks only			[10]
	Calculate value of z using width of interval:	$z = 1{\cdot}82 / 2s = 0{\cdot}91 / 0{\cdot}5128$		
	(allow M1 for $z = 1{\cdot}82 / s$)	$= 1{\cdot}77[46]$	M1 A1	
	Find $\Phi(z)$ and value of α :	$\Phi(z) = 0{\cdot}962 = 1 - ½ (1 - \alpha/100)$		
	(overlooking % is misread;	$\alpha = 100 \{1 - 2 (1 - 0{\cdot}962)\}$		
	M0 for $\alpha = 96{\cdot}2$ or finding $100 - \alpha$)	$= 92{\cdot}4$	M1 A1	[4]