FURTHER MATHEMATICS

Paper 9231/11
Paper 11

General Comments

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There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

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Comments on Specific Questions

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This question was well done and so candidates got off to a good start. Nearly all candidates realised that they needed to sum the series using the method of differences. A small number found $\sum_{r=1}^{n} u_r$ and then most of these candidates used $\sum_{r=13}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{12} u_r$ to obtain the correct result. When small slips occurred, the mark for the sum to infinity could often be earned on a follow through basis.

Answers: $\frac{1}{5} - \frac{1}{\sqrt{2n+1}}$, $\frac{1}{5}$

Question 2

This question proved to be very accessible to the vast majority of candidates. Only a very small number made errors with the initial differentiation. A small number then tried, in vain, to use $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. The remainder knew the appropriate form for parametric equations and generally obtained the correct answer, except for a small proportion who made errors with trigonometric formulae or errors involving the square root.

Answer: $\sqrt{2}\left(e^{\frac{1}{2}\pi} - 1\right)$ (= 5.39)

Question 3

The vast majority of candidates earned the first two marks and were then able to conjecture an expression for S_n successfully. The candidates who thought that $S_n = r \times r!$ were not able to progress, but those who had the correct conjecture were able to produce a correct proof by mathematical induction in a very satisfactory manner.

Answers: 1, 1, 1, 1; $S_n = (n + 1)! - 1$

Question 4

Nearly all candidates were able to find the equations of the asymptotes of C correctly. In the second part of the question the majority considered the sign of the discriminant of a relevant quadratic equation in x with coefficients in terms of y. This led them to conclude that there were no points on C for which $(y - 1)(y - 9) < 0$. The final mark could only be awarded to those who produced a thumbnail sketch, or equivalent, to show how the inequality was solved, since the answer was a given result. A substantial minority attempted the second part of the question using differentiation, in order to find a local maximum and a local minimum. Those who did not produce an appropriate test for each, and did not consider the discontinuity at $x = 1$, lost marks.

Answers: $x = 1$, $y = 2x + 3$

Question 5

In the first part of the question those who considered the determinant equal to zero, in order to find the value of a, invariably gained all three marks. Those who tried to work with equations were far more likely to score one or two marks as their working was either insufficient, or muddled, or both. In part **(i)**, either using row reduction of an augmented matrix, or by working with equations, many obtained $x - y + 2z = 4$ and $5z = 9$ but did not go further and produce a general solution. In part **(ii)**, only a handful of candidates recognised the correct geometrical interpretation of the situation.

Answers: $a = -1$, **(i)** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \\ 1.8 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (oe), **(ii)** planes form a prism (oe)

Question 6

The first part of the question was done extremely well by many candidates. The majority used the binomial expansion of $(c + is)^5$ and equated the real part to $\cos 5\theta$, whereupon the use of $\cos^2\theta = 1 - \sin^2\theta$ led to the correct result. A small minority used $(z + z^{-1})^n = 2\cos n\theta$, where $z = \cos\theta + i\sin\theta$, but were seldom able to complete the proof successfully. In the second part of the question, candidates who showed insufficient reasoning in reaching the displayed result lost one or two marks. It was necessary to show the solutions of $\cos 5\theta = 0$ and to explain how to match one of these solutions to the value $\frac{3-\sqrt{5}}{8}$ which had been obtained by solving the quadratic equation in $\sin^2\theta$.

Question 7

Nearly all candidates were able to use integration by parts to obtain the given reduction formula correctly, with only a small minority showing insufficient working or making a sign error. Most then went on to use this reduction formula. A sizeable number of candidates made errors in finding I_0, or I_1, with the result that their value for I_4 was incorrect. Those who obtained the correct value could easily see that $0 < I_4 < 1$ led to the displayed result, usually earning two of the three marks. The third mark was awarded to those who provided a correct sketch of the function in the interval (0,1) or who explained, clearly, how the function behaved in this interval.

Answer: $I_4 = 24e - 65$

Question 8

Nearly all candidates gained the first three marks for sketching the circle and the cardioid correctly on the same diagram. A surprising number of candidates wrote the coordinates of the points of intersection in the wrong order, thereby losing one of the two marks. Others gave values for the angle outside the range specified in the question, while still others gave a mixture of Cartesian and polar coordinates, writing 0 when they should have written a. Candidates usually identified correctly the region whose area was to be determined. Inevitably some lost accuracy marks because of incorrect limits for the integration, sign errors with the trigonometry, or failure to add half the area of the circle to the result of their integration.

Answers: $\left(a, \frac{\pi}{2}\right)$, $\left(a, \frac{3\pi}{2}\right)$

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The majority of candidates adopted the easier approach in the first part of the question. This involved finding $\frac{dv}{dx}$ and $\frac{d^2v}{dx^2}$ from $v = xy$, rather than finding $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from $y = vx^{-1}$, before substituting in the differential equation to demonstrate the required result. Large numbers of candidates found correctly the complementary function and particular integral. Most of them found the correct general solution for y in terms of x, but a small number stopped at v in terms of x.

Answer: $y = \frac{1}{x}\left(Ae^{-3x} + Be^{x} + 2e^{2x}\right)$

Question 10

Much good work was demonstrated in all three parts of this question. In part **(i)**, a surprising number of candidates did not produce the most direct method of solution, preferring to find a two parameter expression for the vector $\overrightarrow{PQ}$. They then found a pair of simultaneous equations by equating to zero the scalar product formed between $\overrightarrow{PQ}$ and each direction vector of the lines l_1 and l_2. These equations had awkward coefficients and even more awkward solutions. Nevertheless, most candidates' algebra and arithmetic was sufficiently good to emerge with the correct answer. The preceding method sometimes appeared in part **(ii)** as a means of getting the direction of the line PQ, when it had not been found in part **(i)**. In part **(iii)** some candidates used the formula for the distance of a point from a line, while other candidates used the triple scalar product method.

Answers: **(i)** $\frac{4}{\sqrt{3}}$ (= 2.31), **(ii)** $4x + 5y + z = -15$ (oe), **(iii)** $\frac{38}{\sqrt{42}}$ (oe) (= 5.86)

Question 11 Either

Those choosing this alternative did the preliminary parts extremely well, apart from the occasional sign error. In general the required method was known for each of the four parts. In the final part of the question, most candidates realised that the substitution $x = y - 1$ was needed and a considerable number obtained the correct quartic equation $y^4 - 4y^2 + 4 = 0$. Most who got to this stage were able to solve for y correctly. A substantial minority then did not find x. A large number of candidates did not indicate that a quartic equation has four roots, in this case two pairs of equal roots, thus losing the final mark. Rather than pointing out the repeated roots, a small number of candidates earned the final mark by finding the product of two linear factors from the two roots which they had obtained, then dividing the quartic expression by this product and factorising the quotient.

Answers: **(i)** –4, **(ii)** 12, **(iii)** 4, **(iv)** 12, $x = \pm\sqrt{2} - 1$ (twice)

Question 11 Or

Those who chose this somewhat less popular alternative generally produced good work. Some lost several of the early marks, particularly on the first small proof, usually by producing an incomplete argument. The second small proof was generally done rather better, but again some arguments skipped a stage or two. A number of candidates did not appreciate that, with zeros below the leading diagonal, the eigenvalues of the matrix **A** could immediately be written down. They usually obtained the correct values from the characteristic equation or by some other factorisation method. Eigenvectors were generally found accurately, thus giving the matrix **P**. Most candidates deduced the eigenvalues for **A** + 3**I** correctly. Weaker candidates thought that these were the eigenvalues for **B**, but most found the reciprocals of them and correctly wrote the matrix **D**.

Answers: $\mathbf{P} = \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 25 \\ 0 & 0 & 20 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

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FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key Messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

General Comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was an unusually strong preference for the Statistics option, though the small number of candidates who chose the Mechanics option frequently produced equally good attempts. Indeed all questions were answered well by some candidates, with **Questions 2**, **4**, **6** and **10** found to be the most challenging.

Advice to candidates in previous reports was seemingly heeded by many: to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions, as in **Question 4**, and the directions of motion of particles, as in **Question 1**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Question 11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on Specific Questions

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FURTHER MATHEMATICS

Paper 9231/22

Paper 22

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Answer: **(i)** $y = 0{\cdot}388x + 3{\cdot}57$; **(ii)** 0·582; **(iii)** 7·45; **(iv)** $N > 16$.

Question 10

When integrating f(x) to find the distribution functions F of X for $1 \leqslant x \leqslant 3$ and hence G of Y, candidates should remember to include an appropriate constant of integration to ensure that F(1) = 0 and F(3) = 1. For completeness, G(y) should also be stated for values of y outside the central interval of $1 \leqslant y \leqslant 27$, and it should similarly be made clear on the graph of Y's probability density function g that it is $\frac{1}{6y^{\frac{2}{3}}}$ within this central interval and zero outside it. In the final part, the mean of Y is found in the usual way by integrating y g(y) over $1 \leqslant y \leqslant 27$, giving a value of 10. Candidates can save time here by not finding the median value 8, since the required probability can be found from G(10) – ½ rather than G(10) – G(8).

Answer: 0 ($y < 1$), $\frac{1}{2}(y^{\frac{1}{3}} - 1)$ ($1 \leqslant y \leqslant 27$), 1 ($y > 27$); 0·0772.

Question 11 (Mechanics)

This optional question was attempted by only a very small proportion of the candidates, but most of those who did so made good attempts at the first part. One possible method of solution involves using the parallel and perpendicular axes theorems as appropriate to find the moments of inertia of each of the four rings about AB, then summing the results. Alternatively the parallel axes theorem can be used as appropriate to find the moments of inertia of each of the rings about an axis through the centre of mass of the composite object and perpendicular to its plane. After summing the four results, application of the perpendicular axis theorem gives the required moment of inertia about AB. In the second part, candidates should realise that the addition of the particle at D increases the moment of inertia of the body to $I = 14(2 + \sqrt{3})mr^2$. The required angular speed ω then follows from applying conservation of energy to give $\frac{1}{2}I\omega^2 = 9mg \times R\sin 60°$.

Answer: **(ii)** $\sqrt{\frac{9r}{14r}}$.

Question 11 (Statistics)

The required 95% confidence interval was found correctly by most candidates, using the unbiased estimate 2·549 of the population variance for A and a tabular t-value of 2·365. The most appropriate assumptions in the second part are that the distribution of the lengths is also normal for lake B, and that its population variance is the same as that for A's distribution. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. The unbiased pooled estimate 2·436 of the common population variance may be used to calculate a t-value of 0·712. Since it is a one-tail test, comparison with the tabulated value of 1·782 leads to acceptance of the null hypothesis, namely that the mean length of fish in A is not greater than that of fish in B. The final confidence interval is found in the usual way with the same estimate of the common variance but with a tabular t-value of 2·179.

Answer: [11·9, 14·6]; [–1·24, 2·44].