FURTHER MATHEMATICS

Paper 9231/11
Paper 11

General Comments

The scripts for this paper were of a generally good quality. There were a substantial number of high quality scripts and many showing evidence of sound learning. Work was well presented by the majority of candidates. Solutions were generally set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. Candidates tackled the questions on the area within a polar graph, summation of a series by the difference method, eigenvalues and eigenvectors, differential equations, arc length and surface of revolution confidently. The questions on proof by induction and complex numbers showed better levels of understanding and presentation.

Comments on specific questions

Question 1

This question was well done by nearly all candidates, so most got off to a confident start.

Answer: 6π .

Question 2

The answers to this question exhibited a better understanding of proof by induction than has often been the case in the past. Marks were lost by not stating an inductive hypothesis correctly and thinking that 'for all $n \geq 1$' meant 'for all positive integers', when stating a conclusion. The proof that $P_k \Rightarrow P_{k+1}$ was generally sound and was done in a variety of ways.

Question 3

The value of c was invariably stated correctly, with only the very weakest candidates making a sign error. Candidates usually made the correct substitution in the cubic equation, but a substantial number did not simplify the equation to the standard form of a cubic equation. This caused problems for further progress with the question. Having obtained a simplified cubic equation in y, the better candidates saw that a substitution of $y = z^{-1}$ was required to find the cubic equation from which they could deduce the value of $\sum \frac{1}{(\alpha+\beta)^2}$.

Answers: 2; $y^3 - 4y^2 + y + 2 = 0$; $2z^3 + z^2 - 4z + 1 = 0$; $4\frac{1}{4}$.

Question 4

There were two approaches to this reduction formula. The more common one was to integrate by parts, taking unity as one part. The alternative approach was to first find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1+x^2\right)^{-n}\right)$ and then integrate between 0 and 1. The derivation of the reduction formula was difficult for the weaker candidates. They did, however, score well in the final calculation of I_3.

Answer: $\frac{1}{4}+\frac{3}{32}\pi$.

Question 5

The vast majority were able to use the method of differences to sum the given series. Many then went on to find the sum from $N+1$ to $2N$ correctly. Those who decided to express this sum with a common denominator were often able to deduce the final inequality.

Question 6

This question on eigenvalues and eigenvectors was generally done well by many candidates. The result $\mathbf{Ae} = \lambda\mathbf{e}$ was used to show that $\mathbf{e}$ was an eigenvector and to deduce the corresponding value of λ. Most were then able to find the characteristic equation and the remaining two eigenvalues, or reduce the matrix to echelon form by row operations, from which the eigenvalues could be clearly seen. Candidates were usually able to show that $\mathbf{e}$ was an eigenvector of $\mathbf{B}$ with eigenvalue 3, but only the better candidates could show $\mathbf{ABe} = \mu\mathbf{e}$ and determine the value of μ.

Answers: 2; –1, 1; $\mathbf{e}$, 6.

Question 7

Most candidates appreciated that they needed to expand $\left(z-\frac{1}{z}\right)^6$ using the binomial theorem and then group terms. The most common errors, thereafter, concerned the value of $(2\mathrm{i}\sin\theta)^6$, where some thought it to be positive and others had a coefficient of 32, rather than 64. Consequently values of p, q, r and s had the wrong sign, or were double what they ought to have been, or both of these. Most candidates made reasonable attempts at the integration, earning the method marks, if they had been inaccurate earlier on. There were a pleasing number of completely correct solutions.

Answers: $p = 10$, $q = -15$, $r = 6$, $s = -1$; $\frac{5\pi}{64}-\frac{11}{48}$ (OE).

Question 8

Most candidates made a reasonable attempt at reducing both matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ to echelon form, from which they could find bases for K_1 and K_2. The basis vectors were sometimes found inaccurately. The second part of the question proved to be difficult for many. They did not appreciate that $\mathbf{x}_1$ and $\mathbf{x}_2$ could be expressed in the form

$\mathbf{x}_1 = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}$, from which $\mathbf{x}_1-\mathbf{x}_2 = \begin{pmatrix}\lambda-\mu\\\lambda-2\mu\\2\lambda-\mu\\-\lambda+\mu\end{pmatrix}$. Equating this to the form given in the question then enabled the values of λ and μ to be found, and thus the values of p and q also. Only the very best candidates obtained these results correctly.

Answers: $\left\{\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}\right\}$; $p = 4$, $q = -4$.

Question 9

This question was answered extremely well by most candidates, indicating that they had been well taught. The complementary function was found correctly, despite the auxiliary equation having repeated roots. An appropriate particular integral was usually found, so that most had a correct general solution. Some inaccuracies occurred for the weaker candidates, when finding the values of the arbitrary constants, although the method was invariably sound. A correct limit was often found when previous errors occurred, thus resulting in the award of a follow through mark.

Answers: $x = \mathrm{e}^{-\frac{t}{2}} + 3t\mathrm{e}^{-\frac{t}{2}} + \frac{2}{3}\mathrm{e}^{-2t}$, $\lim\limits_{t \to \infty} x = 0$.

Question 10

Almost all candidates were able to write down the equations of the asymptotes of *C*. The second part of the question was where most marks were lost. An analysis, using the discriminant of the quadratic equation in *x*, where coefficients were expressed in terms of *y*, was the most usual approach, but inequalities were apt to go wrong and terms of the quadratic to have the wrong sign. There were good, and often successful, attempts to find the coordinates of the stationary point on *C*. Sketches of *C* were often substantially correct. A mark was commonly lost by not finding the intersection of *C* with the horizontal asymptote.

Answers: $x = 1$, $y = 2$; $\left(7, \frac{25}{12}\right)$; (-0.5,0), (2,0), (0,–2) and (4,2).

Question 11 Either

The candidates attempting this alternative felt confident in their knowledge of the secant function and generally scored most of the marks, with a good number scoring full marks. A correct integral representation of the arc length of *C* was readily obtained using the appropriate results for secants. The evaluation of the arc length was also straightforward for most. The surface area formula was well known and the result for *S* followed. The differentiation in part **(ii)** caused little difficulty and most candidates saw its relevance in evaluating *S*, usually correctly.

Answers: $\left[2 - \frac{1}{4}\pi\right]$, $4\pi\sqrt{2}$.

Question 11 Or

There was much good work evident in this vector question. The first part was usually done by finding the normal vector to the plane Π_1. Candidates then knew the form of the cartesian equation and were able to find the constant term, by using one of the known points in the equation. The middle section proved the most difficult part of the question. The area of the triangle *ABC* was usually correctly found from the magnitude of an appropriate vector product. Some then became confused as to which length constituted the perpendicular height of the tetrahedron. The most successful candidates used the distance of a point from a plane formula to find the distance of *D* from Π_1. The formula given in the question could then be applied to find the volume. Others knew the formula $\left|\frac{1}{6}\mathrm{a.}\,(\mathrm{b} \times \mathrm{c})\right|$ and were able to apply it successfully to this problem.

In the final part, most knew that they required the normal vector to Π_2 and could then use the scalar product to find the angle between the normals to Π_1 and Π_2.

Answers: $3x - y - 6z = 17$; $\frac{13}{3}$ (OE); 40.5º.

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

General Comments

The scripts for this paper were of a generally good quality. There were a substantial number of high quality scripts and many showing evidence of sound learning. Work was well presented by the majority of candidates. Solutions were generally set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. Candidates tackled the questions on the area within a polar graph, summation of a series by the difference method, eigenvalues and eigenvectors, differential equations, arc length and surface of revolution confidently. The questions on proof by induction and complex numbers showed better levels of understanding and presentation.

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This question was well done by nearly all candidates, so most got off to a confident start.

Answer: 6π .

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The answers to this question exhibited a better understanding of proof by induction than has often been the case in the past. Marks were lost by not stating an inductive hypothesis correctly and thinking that 'for all $n \geq 1$' meant 'for all positive integers', when stating a conclusion. The proof that $\mathrm{P}_k \Rightarrow \mathrm{P}_{k+1}$ was generally sound and was done in a variety of ways.

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The value of c was invariably stated correctly, with only the very weakest candidates making a sign error. Candidates usually made the correct substitution in the cubic equation, but a substantial number did not simplify the equation to the standard form of a cubic equation. This caused problems for further progress with the question. Having obtained a simplified cubic equation in y, the better candidates saw that a substitution of $y = z^{-1}$ was required to find the cubic equation from which they could deduce the value of $\sum \frac{1}{(\alpha + \beta)^2}$.

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There were two approaches to this reduction formula. The more common one was to integrate by parts, taking unity as one part. The alternative approach was to first find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1 + x^2\right)^{-n}\right)$ and then integrate between 0 and 1. The derivation of the reduction formula was difficult for the weaker candidates. They did, however, score well in the final calculation of I_3.

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Answers: 2; –1, 1; **e**, 6.

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Answers: $p = 10$, $q = -15$, $r = 6$, $s = -1$; $\frac{5\pi}{64}-\frac{11}{48}$ (OE).

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Answers: $x = e^{-\frac{t}{2}} + 3te^{-\frac{t}{2}} + \frac{2}{3}e^{-2t}$, $\lim\limits_{t \to \infty} x = 0$.

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Almost all candidates were able to write down the equations of the asymptotes of C. The second part of the question was where most marks were lost. An analysis, using the discriminant of the quadratic equation in x, where coefficients were expressed in terms of y, was the most usual approach, but inequalities were apt to go wrong and terms of the quadratic to have the wrong sign. There were good, and often successful, attempts to find the coordinates of the stationary point on C. Sketches of C were often substantially correct. A mark was commonly lost by not finding the intersection of C with the horizontal asymptote.

Answers: $x = 1$, $y = 2$; $\left(7, \frac{25}{12}\right)$; (-0.5,0), (2,0), (0,–2) and (4,2).

Question 11 Either

The candidates attempting this alternative felt confident in their knowledge of the secant function and generally scored most of the marks, with a good number scoring full marks. A correct integral representation of the arc length of C was readily obtained using the appropriate results for secants. The evaluation of the arc length was also straightforward for most. The surface area formula was well known and the result for S followed. The differentiation in part **(ii)** caused little difficulty and most candidates saw its relevance in evaluating S, usually correctly.

Answers: $\left[2 - \frac{1}{4}\pi\right]$, $4\pi\sqrt{2}$.

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In the final part, most knew that they required the normal vector to Π_2 and could then use the scalar product to find the angle between the normals to Π_1 and Π_2.

Answers: $3x - y - 6z = 17$; $\frac{13}{3}$ (OE); 40.5º.

FURTHER MATHEMATICS

Paper 9231/13
Paper 13

General Comments

The scripts for this paper were of a generally good quality. There were a substantial number of high quality scripts and many showing evidence of sound learning. Work was well presented by the majority of candidates. Solutions were generally set out in a clear and logical order. This year a significant minority of candidates did not show sufficient working to obtain full marks for some questions, especially where the answer was a displayed result. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. Candidates tackled the questions on summation of series, implicit differentiation, matrices, differential equations and vectors confidently. The question on proof by induction continued to show better levels of understanding and presentation.

Comments on specific questions

Question 1

Most candidates were able to simplify the given expression and many were able to complete the question successfully. A small number of candidates lost the final mark by leaving their answer in the form f(2*n* +1) – f(*n* + 1), while some failed to spot the pattern of the cancelling terms.

Answers: $r!(r^2+1)$; $2n(2n+1)! - n(n+1)!$.

Question 2

This question caused many problems especially in the first part where too many did not realise that, in order to eliminate the square root of *y*, they needed to isolate the term in $\sqrt{y}$ before squaring. Others did not show sufficient working to obtain the displayed quartic equation. In the second part many candidates correctly obtained the given result, however, some were not aware that the result could be obtained directly, without calculating all the values of S_n for n = 1,2,……,8. Much time was wasted by this calculation approach, sometimes with little success.

Question 3

The method of proof by induction was understood reasonably well by many candidates. Marks were lost for showing insufficient working in deducing $e^x \sin x + e^x \cos x = \sqrt{2}e^x\, sin\left(x + \frac{1}{4}\pi\right)$ and

$$\left(\sqrt{2}\right)^k\left(\sin\left(x+\frac{1}{4}k\pi\right)e^x + e^x\cos\left(x+\frac{1}{4}k\pi\right)\right) = \left(\sqrt{2}\right)^{k+1}e^x\sin\left(x+\frac{1}{4}(k+1)\pi\right).$$

The final mark was sometimes lost for an imprecise conclusion, notably by thinking that ‘for all $n \geq 1$’ meant ‘for all positive integers’.

Question 4

This question on implicit differentiation was generally well done. Nearly all candidates correctly differentiated the given equation to obtain the value of y' at the point $A(1,–2)$. The majority then successfully differentiated again to obtain the value of y'' at $A(1,–2)$. A common error in the second derivative was to write $6y(y')^2$ as $6yy'$.

Answer: $y'' = -\frac{20}{27}$.

Question 5

The initial integral was invariably done correctly, mostly by inspection, but some used a substitution. A few candidates incorrectly thought, in both of the first two parts, that $\int e^{-x^2}\,dx = -\frac{e^{-x^2}}{2x} + c$. The majority had more success in obtaining the value of I_7 although quite a significant minority wrote the final answer as 8e.

Answer: $\frac{8}{e}$.

Question 6

This question on linear transformations was successfully completed by a sizeable minority of candidates. The row reduction of matrix **M** was generally well done, and the majority of candidates established a correct basis for K_2, using a valid method, in the case $\alpha = 0$. However, only the better candidates obtained a correct basis for K_1 in the case $\alpha \neq 0$. In this case, weaker candidates often failed to recognise that dimension $K_1 = 1$.

Answers: $K_1 \left\{\begin{pmatrix}25\\10\\1\\3\end{pmatrix}\right\}$; $K_2 \left\{\begin{pmatrix}23\\8\\2\\0\end{pmatrix}, \begin{pmatrix}9\\4\\0\\2\end{pmatrix}\right\}$.

Question 7

This question was answered well by a large number of candidates. Most candidates were able to correctly differentiate the suggested particular integral twice and thus establish the appropriate value of λ. A correct complementary function was usually found. Thus the vast majority obtained a correct general solution. This was correctly differentiated and the values of the arbitrary constants were found, using the initial conditions. A significant number of weaker candidates ignored the first part of the question and sought a particular integral of the form Ce^{-x}, leading to an inconsistency which they were unable either to recognise or resolve.

Answers: $\lambda = 2$; $y = 3e^{-x} - e^{-4x} + 2xe^{-x}$.

Question 8

Most candidates were able to obtain a correct integral representation for the arc length. Some candidates were unable to perform the required integration. A small number of these candidates evaluated the integral using their graphics calculator, which is forbidden in the instructions on the front of the question paper. Many were able to obtain the correct arc length by integrating correctly. The coordinates of the centroid caused problems for the large proportion of candidates who thought that, for example, $\int y\mathrm{d}x$ was $\int t^3\mathrm{d}t$ in place of $\int t^3.3t\mathrm{d}t$. Those who overcame this difficulty, by working in the *x*-domain, frequently forgot to adjust the limits of integration from 0 to 2 to 0 to 6. This earned method marks, but not accuracy marks, if everything else was correct. Those who overcame these problems invariably ended up with the correct coordinates for the centroid.

Answers: $5\sqrt{5}-1$ or 10.2; $\left(\frac{30}{7},\frac{5}{2}\right)$ or (4.29, 2.5).

Question 9

In the first part of the question **Ae =** λ**e** was frequently stated, but only the better candidates were able to establish, rigorously, that **Me** was an eigenvector of the matrix **B**. Thus many only scored one of the first three marks. Sadly, a rather large number of candidates had to find the eigenvalues of the matrix **A** from its characteristic equation, ignoring the instruction to 'write down the eigenvalues of **A**'. This wasted valuable time for them. The more alert candidates noted that the matrix was given in echelon form and looked at the leading diagonal. Good attempts were made to find the eigenvectors. The majority of candidates used the result from the first part of the question to find the eigenvalues and eigenvectors of **B**, but a small minority again found the characteristic equation in order to obtain eigenvalues and then used them to calculate eigenvectors. Only the very weakest candidates said that the eigenvectors of **B** were the same as those of **A**.

Answers: Eigenvalues are: –1, 1, 2 for **A** and **B;** Eigenvectors of **B** are $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}4\\4\\1\end{pmatrix}$ respectively.

Question 10

Few candidates gained full marks for this question. In the first part, nearly all were able to score the first method mark and then failed to point to the sine-cosine link and just wrote down the given result. When an answer is displayed full working is required. Many were able to draw an adequate sketch of the polar graph. A fair proportion of these candidates were able to state the equation of the line of symmetry, but answers omitting 'θ =', or worse still, replacing 'θ' by '*r*' were not awarded the mark. Only a small minority were able to give an acceptable reason for the equation of the line of symmetry, either by indicating that *r* reached its maximum value when the cosine function attains its maximum value, or $\frac{\mathrm{d}r}{\mathrm{d}\theta}=0$. Good attempts were usually made to find the area of the region enclosed by the curve, but here again, at the end of the question, one or two marks could be lost by jumping straight to the displayed answer.

Answer: $\theta=\frac{3}{8}\pi$.

Question 11 Either

This was by far the more popular alternative. There were a good number of completely correct answers by efficient methods, showing that candidates had been well taught. In the first part most wrote down a general vector for $\overrightarrow{PQ}$ and then used orthogonality conditions to find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Some obtained the common normal, $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, to the lines l_1 and l_2 first and then wrote $\overrightarrow{PQ}$ as a multiple of this to obtain $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. A substantial number of candidates trying this approach lost marks, however by trying to equate $\overrightarrow{PQ}$ with **n** itself, rather than a multiple of **n**. Candidates obtaining bizarre components in their answers for $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ should have checked their working to see if they could find an error. Often it was a wrong sign or coefficient. In the second part, the best candidates quickly found the common perpendicular to the lines *AB* and *PQ* and then found the projection of $\overrightarrow{AP}$ (or equivalent) onto this normal in order to find the shortest distance between them. Weaker candidates were unable to identify the distance that was required in this last part of the question and some simply found the length of *PQ*.

Answers: $\mathbf{p} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$; 5.52.

Question 11 Or

Most candidates, of the small number attempting this alternative, were able to show the cube roots of unity on an Argand diagram and show that the square of one non-real cube root gave the other. They were also able to express these non-real cube roots in cartesian form. Most were also able to evaluate the determinant in terms of ω, without obtaining its numerical value. Only the best candidates were able to express z in polar form and hence obtain the cube roots of z.

Answers: –18; $8\left(\cos\frac{7}{6}\pi + \mathrm{i}\sin\frac{7}{6}\pi\right)$, $2\left(\cos[7+12k]\frac{\pi}{18} + \mathbf{i}\sin[7+12k]\frac{\pi}{18}\right)$ $k = 0, 1, 2.$

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram.

General comments

Most candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics option, though some of the small minority of candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1**, **3**, **5** and **6**. The second part of **Question 4** was found to be challenging by many candidates.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 1** and the directions of motion of particles as in **Question 2**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 5** and **9** any symbols used for variables when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, some rounded intermediate results to this same accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

The essential requirement is the ratio of the friction F and reaction R at A, and candidates found this in a variety of ways, often involving also the tension T in the string. It is also necessary to use the sine and cosine of a chosen angle, for example that which the rod makes with the ground and which equals the angle *APC*. Candidates should not use the value of the angle itself since this introduces a slight inaccuracy in the final result, but instead note that $PC = 4a$ and thus produce exact values 3/5 and 4/5 for the sine and cosine. One popular approach was to take moments about *A* for the rod which yields *T* immediately, and then horizontal and vertical resolution of forces on the rod give *F* and *R* respectively, though they need not in fact be found explicitly since it is their ratio which is required. Alternatively moments about *P* and *C* for the rod avoid the introduction of *T*.

Question 2

The first part of the question requires three equations to be formulated, using conservation of momentum, Newton's restitution equation and the fact that *A* loses three-quarters of its kinetic energy in the collision with *B*. The first two of these presented few problems, but candidates should not introduce the subsequent speed of *B* into their kinetic energy equation since it concerns *A* only. One of the possible methods of solution is to combine the first two equations and obtain the speed v_A after the collision in the form $\frac{1}{3}(2 - e)u$. Inserting this in place of v_A in the kinetic energy equation gives a quadratic equation for *e*, and candidates should produce both roots ½ and 3½ before explicitly rejecting the latter on the grounds that it exceeds unity. The alternative approach of noting from the kinetic energy equation that $v_A^2 = \frac{1}{4}u^2$ led many candidates to consider only the case $v_A = \frac{1}{2}u$ but not $v_A = -\frac{1}{2}u$, thereby overlooking the possible value 3½ of *e*. The second collision is rather simpler in that only two equations are required, derived from conservation of momentum and the restitution equation, from which the speed of *B* after the second collision is found to be $\frac{3}{4}u$. Since this exceeds the speed $\frac{1}{2}u$ of *A* no further collision between these spheres is possible. Candidates should note that it is unnecessary to find the corresponding speed *u* of *C* since the spheres *B* and *C* will necessarily move apart after their collision.

Answers: $\frac{3}{4}u$.

Question 3

Many candidates successfully formed two equations by using conservation of energy during the motion from the initial point *O* and by equating the net radial force to mv^2/a, and then eliminated the speed *v* of the particle to produce the required expression for the tension *T*. This yields the maximum and minimum values of *T* when θ is taken to be respectively 0 and π (but not $\frac{1}{2}\pi$ as was sometimes seen) and equating the ratio of these values to 3 gives *u*. Many candidates also realised that in the final part the earlier expression for *T* should be equated to $\frac{1}{2}(mu^2/a + mg)$, and substitution for *u* yields the required value of $\cos\theta$.

Answers: $\sqrt{(8ag)}$; $-\frac{1}{2}$.

Question 4

The moments of inertia about their centres of the sphere, rod and ring are first found, utilising if necessary the *List of Formulae* for the first two, and use of the parallel axis theorem yields their moments of inertia about the specified axis at *C*. Summation then produces the given expression for the system, which most candidates obtained successfully. In questions such as this in which a given result must be shown, candidates are advised to do rather more than simply write down the sum of up to six terms with no explanation. Finding the value of λ in the second part proved particularly challenging, with many candidates resorting fruitlessly to considerations of energy and sometimes confusing their symbol ω for angular speed with the same symbol in the SHM equation. The specification of small oscillations in the question should perhaps suggest that the product of the moment of inertia and the radial acceleration be equated to the couple acting on the system when it is displaced a small angle θ from the vertical. Approximating $\sin\theta$ by θ leads to an equation in standard SHM form $d^2\theta/dt^2 = -\omega^2\theta$. Finally the period given in the question is equated to $2\pi/\omega$ in order to produce the value of λ.

Answers: $\lambda = 3/2$.

Question 5

Candidates should begin with an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho < 0$, and in doing so should be aware that *r* and ρ are not the same entity. Comparison of the given product moment correlation coefficient for the sample with the tabular value 0·497, taking particular care since the former is negative, leads to a conclusion of there being no evidence of negative correlation.

Question 6

While a few candidates mistakenly identified the distribution of X as being Poisson, most correctly stated it to be a negative exponential and gave the correct mean. P(X > 4) was also usually evaluated correctly from 1 – F(4), as was the median m by solving F(m) = ½. When giving non-exact numerical answers, candidates should recall the requirement that these be correct to 3 significant figures, so that 0·091 for example is insufficiently accurate.

Answers: 5/3 or 1·67; 0·0907; (5/3) ln 2 or 1·16.

Question 7

Finding the required expected frequencies was usually done correctly, involving the integration of f(x) with appropriate limits and multiplication by the number of observations, here 80. The statements of the hypotheses by a few candidates suggested they were unsure as to what a goodness of fit test actually tests, but most correctly stated the hypotheses in terms of whether or not the given function f fits the observations. Comparison of the calculated χ^2-value 5·7 of with the tabular value 6·25 shows that f does indeed fit.

Answers: 20, 12, 8.

Question 8

Showing that the function g has the required form essentially requires that the cumulative distribution function F of X be found by integration, and hence the corresponding function G of Y, followed by differentiation to give g. Most candidates performed these operations correctly for $8 \le y \le 64$, but the cases of $y < 8$ and $y > 64$ should not be overlooked. E(Y) was usually found correctly by integrating yg(y) with respect to y, though it can equally well be derived from E(X^3).

Answers: 496/15 or 33·1.

Question 9

As in all such tests, the hypotheses required in both parts of this question should be stated in terms of the population means and not the sample means. In the first part the unbiased estimate 119/90 or 1·322 of the population variance may be used to calculate a t-value of 2·48 (2·47 is acceptable). Since it is a one-tail test, comparison with the tabulated value of 1·833 leads to acceptance of the alternative hypothesis, namely that the population mean is greater than 5·2 and the gardener P's claim is justified. Estimation and appropriate use in the second part of either the pooled variance 1·25 or the common variance 0·25 yields a t-value of 1·8, and comparison with the tabulated value of 1·734 again leads to acceptance of the alternative hypothesis, in this case that gardener Q's trees produce a greater mean mass of fruit. Candidates should state the additional assumption that both distributions have the same variance.

Question 10 (Mechanics)

This optional question was much less popular than the Statistics alternative discussed below, and few good attempts were seen, particularly at the final part. The kinetic energy of the particle after falling a distance ka from A requires the difference of the changes in potential energy and the elastic energy of the string, here $mgka$ and $(3mg/2)(ka – a)^2/2a$. Finding the roots 3 and ⅓ of the quadratic equation $3k^2 – 10k + 3 = 0$ and recalling that $k > 1$ shows that the particle first comes to instantaneous rest at a distance 3a vertically below A, as required. Many of those candidates attempting the final part showed that the given term $\sqrt{(2a/g)}$ represents the time taken by the particle to fall a distance a from A to the point at which the string becomes taught. The second given term is more challenging, however, and may be derived by formulating an equation $\mathrm{d}^2x/\mathrm{d}t^2 = – 3gx/2a$ for the subsequent motion, corresponding to SHM with amplitude $x_0 = 4a/3$ about an equilibrium point which is at a distance 5a/3 below A. The required time may then be found from the standard formula $x = x_0 \cos \omega t$ since all the variables except t in it are known.

Question 10 (Statistics)

The two required values of p are found by first formulating an expression for the gradient b_1 of the regression line of y on x, utilising the given data and with possible assistance from the *List of Formulae*, then noting from the equation given in the question that $b = 1$, and thus obtaining a quadratic equation in p which may be solved. The remainder of the question also makes use of standard formulae, namely for the product moment correlation coefficient r and the regression line of x on y. Provided candidates found the correct smaller value of p in the first part, no great difficulty was usually encountered with the following two parts. If candidates choose to find the gradient b_2 of the regression line of x on y from r and b_1 (the latter here unity of course) then they should take care to equate b_1b_2 to r^2 rather than to r.

Answers: 1, 1¾; (**i**) √39/12 or 0·52; (**ii**) $48x = 13y + 65$.

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram.

General comments

Most candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics option, though some of the small minority of candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1**, **3**, **5** and **6**. The second part of **Question 4** was found to be challenging by many candidates.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 1** and the directions of motion of particles as in **Question 2**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 5** and **9** any symbols used for variables when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, some rounded intermediate results to this same accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

The essential requirement is the ratio of the friction F and reaction R at A, and candidates found this in a variety of ways, often involving also the tension T in the string. It is also necessary to use the sine and cosine of a chosen angle, for example that which the rod makes with the ground and which equals the angle *APC*. Candidates should not use the value of the angle itself since this introduces a slight inaccuracy in the final result, but instead note that $PC = 4a$ and thus produce exact values 3/5 and 4/5 for the sine and cosine. One popular approach was to take moments about *A* for the rod which yields *T* immediately, and then horizontal and vertical resolution of forces on the rod give *F* and *R* respectively, though they need not in fact be found explicitly since it is their ratio which is required. Alternatively moments about *P* and *C* for the rod avoid the introduction of *T*.

Question 2

The first part of the question requires three equations to be formulated, using conservation of momentum, Newton's restitution equation and the fact that *A* loses three-quarters of its kinetic energy in the collision with *B*. The first two of these presented few problems, but candidates should not introduce the subsequent speed of *B* into their kinetic energy equation since it concerns *A* only. One of the possible methods of solution is to combine the first two equations and obtain the speed v_A after the collision in the form $\frac{1}{3}(2 - e)u$. Inserting this in place of v_A in the kinetic energy equation gives a quadratic equation for *e*, and candidates should produce both roots ½ and 3½ before explicitly rejecting the latter on the grounds that it exceeds unity. The alternative approach of noting from the kinetic energy equation that $v_A^2 = \frac{1}{4}u^2$ led many candidates to consider only the case $v_A = \frac{1}{2}u$ but not $v_A = -\frac{1}{2}u$, thereby overlooking the possible value 3½ of *e*. The second collision is rather simpler in that only two equations are required, derived from conservation of momentum and the restitution equation, from which the speed of *B* after the second collision is found to be $\frac{3}{4}u$. Since this exceeds the speed $\frac{1}{2}u$ of *A* no further collision between these spheres is possible. Candidates should note that it is unnecessary to find the corresponding speed *u* of *C* since the spheres *B* and *C* will necessarily move apart after their collision.

Answers: $\frac{3}{4}u$.

Question 3

Many candidates successfully formed two equations by using conservation of energy during the motion from the initial point *O* and by equating the net radial force to mv^2/a, and then eliminated the speed *v* of the particle to produce the required expression for the tension *T*. This yields the maximum and minimum values of *T* when θ is taken to be respectively 0 and π (but not $\frac{1}{2}\pi$ as was sometimes seen) and equating the ratio of these values to 3 gives *u*. Many candidates also realised that in the final part the earlier expression for *T* should be equated to $\frac{1}{2}(mu^2/a + mg)$, and substitution for *u* yields the required value of $\cos\theta$.

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Answers: $\lambda = 3/2$.

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Answers: 496/15 or 33·1.

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As in all such tests, the hypotheses required in both parts of this question should be stated in terms of the population means and not the sample means. In the first part the unbiased estimate 119/90 or 1·322 of the population variance may be used to calculate a t-value of 2·48 (2·47 is acceptable). Since it is a one-tail test, comparison with the tabulated value of 1·833 leads to acceptance of the alternative hypothesis, namely that the population mean is greater than 5·2 and the gardener P's claim is justified. Estimation and appropriate use in the second part of either the pooled variance 1·25 or the common variance 0·25 yields a t-value of 1·8, and comparison with the tabulated value of 1·734 again leads to acceptance of the alternative hypothesis, in this case that gardener Q's trees produce a greater mean mass of fruit. Candidates should state the additional assumption that both distributions have the same variance.

Question 10 (Mechanics)

This optional question was much less popular than the Statistics alternative discussed below, and few good attempts were seen, particularly at the final part. The kinetic energy of the particle after falling a distance ka from A requires the difference of the changes in potential energy and the elastic energy of the string, here $mgka$ and $(3mg/2)(ka - a)^2/2a$. Finding the roots 3 and ⅓ of the quadratic equation $3k^2 - 10k + 3 = 0$ and recalling that $k > 1$ shows that the particle first comes to instantaneous rest at a distance $3a$ vertically below A, as required. Many of those candidates attempting the final part showed that the given term $\sqrt{(2a/g)}$ represents the time taken by the particle to fall a distance a from A to the point at which the string becomes taught. The second given term is more challenging, however, and may be derived by formulating an equation $d^2x/dt^2 = - 3gx/2a$ for the subsequent motion, corresponding to SHM with amplitude $x_0 = 4a/3$ about an equilibrium point which is at a distance $5a/3$ below A. The required time may then be found from the standard formula $x = x_0 \cos \omega t$ since all the variables except t in it are known.

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Answers: 1, 1¾; (**i**) √39/12 or 0·52; (**ii**) $48x = 13y + 65$.

FURTHER MATHEMATICS

Paper 9231/23

Paper 23

Key messages

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Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram.

General comments

Most candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was no strong preference for either the Mechanics or the Statistics option and good attempts were seen at both. Indeed all questions were answered well by some candidates, most frequently **Questions 1**, **2, 3**, **7, 8** and **9**. **Question 6** and the last part of **Question 5** were found to be challenging by some but by no means all candidates.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 11** and the direction of motion of particles, as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Question 7** any symbols used for variables when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

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Comments on specific questions

Question 1

Almost all candidates rightly appreciated that the question may be solved by first applying one of the standard formulae for linear motion under constant acceleration to find, for example, the impulse, deceleration or distance and then applying a second formula to find the mass m of the bullet. A small minority either misremembered a formula, writing for example $v = ut + \frac{1}{2}at^2$, or took the change in speed to be 280 + 30 m s^{-1} instead of 280 – 30 m s^{-1}.

Answer: $m = 0{\cdot}06$.

Question 2

The solution may be found by equating two expressions for v^2, where v is the speed of the particle at P. One is found by noting that the normal reaction becomes zero when contact with the surface is lost at P, so that $mv^2/a = mg \cos \theta$, while the second comes from conservation of energy between A and P. Some candidates chose to perform the latter in two stages, between A and B and then B and P, in which case it is essential that the two resulting energy equations are combined to give v^2 at P in terms of u^2.

Answer: $u = \frac{1}{2}\sqrt{(ag)}$.

Question 3

Four equations must be formulated, based on conservation of momentum in the first collision, Newton's restitution equation for each of the two collisions, and the given relationship between the initial and final kinetic energies. In addition to the required coefficient of restitution e between A and B, these equations involve three other unknowns, namely the speeds of A and B after their mutual collision and the speed of B after colliding with the barrier. The latter three unknowns may be eliminated in a variety of ways, yielding the value of e. Perhaps the step requiring the most thought is formulating the kinetic energy equation, where all three terms are required and the mass of B must be taken as $2m$ rather than m.

Answer: $\frac{1}{3}$.

Question 4

The first part of this question was apparently particularly challenging. The starting point is to apply $F = m\,d^2x/dt^2$, where the net force F on the particle is the difference between the two given magnitudes, and care must be taken over the sign of F. Expanding $(1 + x/2a)^{-\frac{1}{2}}$ and neglecting higher order terms yields the SHM equation $d^2x/dt^2 = -(g/4a)x$. Noting that $\omega^2 = g/4a$ and also that the amplitude x_0 is $a/20$, the required period T, speed v and time t are readily obtained from the standard SHM formulae $T = 2\pi/\omega$, $v^2 = \omega^2 (x_0^2 - x^2)$ and $x = x_0 \cos \omega t$. Although the sine may be used instead to find t, care is needed since the motion does not commence from the centre O.

Answer: $4\pi\sqrt{(a/g)}$; $\frac{2}{3}\pi\sqrt{(a/g)}$.

Question 5

Although a very few candidates effectively split the lamina into four component rectangles and found the moment of inertia of each before combining them to yield the required result I_A, it is easier to take the difference between the moment of inertia of the original complete lamina $ABCD$ and that of the lamina $EFGH$ which is removed from it. In all cases the expression for the moment of inertia of a rectangular lamina about a perpendicular axis through its centre may be taken from the *List of Formulae*, taking care to insert the appropriate lengths of the sides, and then the parallel axis theorem applied since the specified axis is at A. Some candidates misread the question as stating that m is the mass of the lamina remaining after the removal of $EFGH$, though this does not of course lead to the given moment of inertia. The second part was answered correctly less often, sometimes simply because the distance AC was taken as $a\sqrt{5}$ rather than $2a\sqrt{5}$. Energy considerations show that the minimum angular speed ω satisfies $\frac{1}{2}I_A\omega^2 = \frac{3}{4}mg \times AC$, from which the minimum value may be found using $u = \omega \times AC$.

Question 6

Very few completely correct answers were seen. Both the required scatter diagrams should show six points, widely and randomly scattered in the first case and as if exactly on a downward sloping straight line in the second. Both the regression lines in the first diagram pass between the points, with y on x approximately horizontal and x on y approximately vertical, while in the second case y on x passes exactly through the six points while x on y should be shown or stated as doing the same. Provided the labelling is clear, candidates need not add any justification or explanation.

Question 7

The key to this question is to calculate the differences between the pairs of observations and base the test on them. It is of course essential to retain the signs of the differences and not just consider their magnitudes. The mean of the resulting sample is then 4/6 and the unbiased estimate of the population variance is 46/15. This gives a calculated value of t of 0·9325, and comparison with the tabulated value 2·015 leads to the conclusion of the mean times over the two courses not being different. Candidates should state the hypotheses explicitly in terms of the population means.

Question 8

Unlike the previous question involving a paired sample, this question requires a two-sample approach. A few candidates treated it as a test of the mean times, whereas it is of course the confidence interval which is required. One valid approach is to estimate the two population variances and hence a combined variance 2·750 from $(55500 - 1752^2/60)/(59 \times 60) + (33500 - 1220^2/50)/(49 \times 50)$ while another is to estimate a common variance 74·8 from $(55500 - 1752^2/60 + 33500 - 1220^2/50) / 108$. In either case the required confidence interval then follows from inserting this estimated variance in the appropriate formula with a critical value of 1·96.

Answer: $4{\cdot}8 \pm 3{\cdot}25$.

Question 9

Almost all candidates found the value of n correctly by equating the usual expression for the gradient in the regression line of y on x to the given value $-\frac{3}{4}$. Similar appropriate expressions, which may be adapted from those for y on x given in the *List of Formulae*, yield the regression line $x = Ay + B$, and the product moment correlation coefficient r may be found either from the expression in the *List of Formulae* or from $\sqrt{(-\frac{3}{4} \times A)}$. Care is needed over the sign with the latter approach. The regression line of x on y cannot of course be found by simply rearranging the equation of the regression line of y on x (with its given gradient $-\frac{3}{4}$). The final regression line may be stated immediately by dividing through $x = Ay + B$ by k, but some candidates did not appreciate this effect of using scaled data.

Answers: **(i)** $n = 4$; **(ii)** $x = -12y/35 + 312/35$; **(iii)** $-3\sqrt{35}/35$; $x' = Ay' + B/k$.

Question 10

Most candidates were able to correctly multiply together the individual probabilities for each of the kicks in the first two parts and simplify their expressions to yield the given probabilities. The probability that Jill wins the game may be found by adding together the probabilities of Jill winning the game on her 1st, 2nd, 3rd, … kick and then summing the resulting arithmetic progression. The probability that the game is a draw may be found in an analogous way, or alternatively by subtracting from unity the probabilities that Jill or Kate wins. In these final two parts, some candidates mistakenly believed that the general term of their arithmetic progression is the required answer.

Answer: **(iii)** ½; **(iv)** 1/6.

Question 11 (Mechanics)

The given relation between tan θ and μ may be found in a variety of ways, though moments for the rod must be taken about at least one point. The centre of the rod probably yields the fastest solution since no other moments or resolution of forces other than horizontally are needed, but otherwise any appropriate combination of moments about A or B or resolution of forces will suffice. The usual limiting friction relation must also be employed at both A and B in order to introduce μ. The final part may be attempted independently of the first part, and the lower bound on μ requires finding the positive solution of the given equation for tan θ with $\theta = 45°$ and thus tan $\theta = 1$. While many candidates did so successfully, the upper bound corresponding to $\theta = 0°$ was found much less frequently.

Answer: $\tfrac{1}{2}\sqrt{6} - 1 \le \mu \le \tfrac{1}{2}\sqrt{2}$.

Question 11 (Statistics)

The first stage is to find 2×2 contingency tables of both the observed and the expected numbers of candidates, and this was usually done correctly. The usual summation for χ^2 here produces four terms, each of which is a constant multiple of $(A - 27)^2$. Given the stated conclusion of independence, this calculated value of χ^2 must be less than the appropriate tabular value 2·706 and thus $(A - 27)^2 < 16{\cdot}07$. While many candidates could find the non-integral bounds on A from this, not all could deduce the corresponding greatest and least possible integer values correctly, taking instead 30 and 24. In the more challenging final part the calculated value of χ^2 is simply increased by a factor N, and if A is given the value 29 then the inequality becomes one for N from which the integral value is found immediately.

Answer: 31, 23; 4.