Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/21**

Paper 2 **May/June 2016**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2016 series for most Cambridge IGCSE®, Cambridge International A and AS Level components and some Cambridge O Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a "fortuitous" answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question	Mark Scheme Details		Part Mark	Total
1 (i)	Find magnitude of impulse by change in momentum: (if 320 + 20, can allow M1 but not MR)	$(\pm)\ 0{\cdot}01 \times (320 - 20)$ $= (\pm)\ 3$ [N s^{-1}]	M1 A1	[2]
(ii)	*EITHER:* Find thickness d by change in energy: (M0 if $\Delta v^2 = 320^2 + 20^2$) *OR:* [Find t from impulse and] use t to find d: (M0 if 320 – 20) *OR:* Find d using $F = ma$, $v^2 = u^2 + 2as$	$1000d = \frac{1}{2} \times 0{\cdot}01 \times (320^2 - 20^2)$ $d = 0{\cdot}51$ [m] $d = \frac{1}{2}(320 + 20)\,t$ $= 170 \times 3/1000 = 0{\cdot}51$ [m] $a = 10^5$, $d = 0{\cdot}51$ [m]	M1 A1 (M1 A1) (M1 A1)	[2]
(iii)	*EITHER:* Find time t from constant deceln. formula: *OR:* Find time t from impulse:	$d = \frac{1}{2}(320 + 20)\,t$, $t = 0{\cdot}003$ [s] $1000\,t = 3$, $t = 0{\cdot}003$ [s]	B1 (B1)	[1]
2	Use momentum <u>and</u> Newton's law, e.g.: (M0 for inconsistent signs; allow $v_A + v_B = u$) Combine to find v_A and v_B (may be implied): Relate speed w_B of B after colln. with wall to v_B, e.g.: Relate KEs before and after (AEF): (Treat error in 1/5 as M1 A0) Find e:	$mv_A + mv_B = mu$, $v_B - v_A = eu$ $v_A = \pm(1 - e)\,u/2$, $v_B = \pm(1 + e)\,u/2$ $w_B = \pm\frac{1}{2}\,v_B$ [$= \pm(1 + e)\,u/4$] $\frac{1}{2}\,mu^2 / 5 = \frac{1}{2}\,mv_A{}^2 + \frac{1}{2}\,mw_B{}^2$ $1/5 = (1 - e)^2/4 + (1 + e)^2/16$ $25e^2 - 30e + 9 = (5e - 3)^2 = 0$ $e = 3/5$ *or* $0{\cdot}6$	M1 A1, A1 B1 M1 A1 M1 A1	[8]
3	*EITHER:* Find time during which speed is low e.g.: (working may be implied) Find ω (implied?) or T from given time: Find other value (√ on first): Find maximum speed v_{max}: *OR:* Find time during which speed is high e.g.: (working may be implied) Find ω (implied?) or T from given time, e.g.: Find other value (√ on first): Find maximum speed v_{max}: *OR:* Find x when speed is $\frac{1}{2}\,v_{max}$: (AEF) Find time during which speed is low, e.g.: *or* speed is high, e.g.: Find ω, T and v_{max} as above	$x = a\cos\omega t$, $v = (\pm)\,a\omega\sin\omega t$ $\frac{1}{2}\,a\omega = a\omega\sin\omega t$ $t_L = \pi/6\omega$ *or* $T/12$ $4t_L = 4/3$, $\omega = \pi/2$ *or* $T = 4$ [s] $T = 2\pi/\omega = 4$ [s] or $\omega = \pi/2$ $v_{max} = 0{\cdot}25\omega = \pi/8$ *or* $0{\cdot}393$ [m] $x = a\sin\omega t$, $v = a\omega\cos\omega t$ $\frac{1}{2}\,a\omega = a\omega\cos\omega t$ $t_H = \pi/3\omega$ *or* $T/6$ $2\pi/\omega - 4t_L = 4/3$, $\omega = \pi/2$ *or* $T = 4$[s] $T = 2\pi/\omega = 4$ [s] or $\omega = \pi/2$ $v_{max} = 0{\cdot}25\omega = \pi/8$ *or* $0{\cdot}393$ [m] $\omega^2(a^2 - x^2) = \frac{1}{4}\,\omega^2 a^2$ $x^2 = \frac{3}{4}\,a^2$, [±] $x = \sqrt{3}a/2$ *or* $\sqrt{3}/8$ $\sqrt{3}a/2 = a\cos(\omega t)$, $\omega t_L = \pi/6$ $\sqrt{3}a/2 = a\sin(\omega t)$, $\omega t_H = \pi/3$	 M1 M1 A1 M1 A1 B1√ M1 A1 (M1) (M1 A1) (M1 A1) B1√ (M1 A1) (B1) (M1 A1)	[8]

Question	Mark Scheme Details		Part Mark	Total
4	Find v^2 at A from conservation of energy:	$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mga(1 + \cos\theta)$	M1 A1	
	Use $F = ma$ radially at A with $R = 0$:	$mv^2/a = mg\cos\theta$	B1	
	Use $v^2 = 3ag/5$ and eliminate $\cos\theta$ to find u:	$\cos\theta = 3/5$ $u^2 = v^2 + 2ag(1 + \cos\theta) = 19ag/5$ $u = \sqrt{(19ag/5)}$ *or* $1{\cdot}95\sqrt{(ag)}$		
	(allow numerical value of g)	*or* $\sqrt{(38a)}$ *or* $6{\cdot}16\sqrt{(a)}$	M1 A1	[5]
	Find or imply vertical component of speed at A:	$V = v\sin\theta\,[= 4v/5]$	M1	
	Find greatest height h_A reached above A:	$h_A = V^2/2g = 24a/125$ *or* $0{\cdot}192a$	M1 A1	
	Find greatest height h_O reached above O:	$h_O = h_A + a\cos\theta$ $= 99a/125$ *or* $0{\cdot}792a$	M1 A1	[5]
5	Find MI of rod about l:	$I_{rod} = \frac{1}{3}\,\frac{3}{4}m(3a/2)^2 + \frac{3}{4}m(\frac{1}{2}a)^2$ $[= \frac{3}{4}ma^2]$	B1	
	Find MI of disc C about l using both theorems:	$I_{disc\,C} = \frac{1}{2} \times \frac{1}{2}ma^2 + m(3a)^2$ $[= (37/4)ma^2]$	M1 A1	
	Find MI of disc D about l using both theorems:	$I_{disc\,D} = \frac{1}{2} \times \frac{1}{2}\,4m(2a)^2 + 4m(3a)^2$ $[= 40\,ma^2]$	M1 A1	
	Sum to find MI of system about l: **A.G.**	$I = (\frac{3}{4} + 37/4 + 40)\,ma^2 = 50\,ma^2$	A1	[6]
	Use eqn of circular motion to find $d^2\theta/dt^2$ where θ is angle of CD with vertical: (A1 only for two correct terms on RHS; A0 if $\cos\theta$ used)	$[-]\,I\,d^2\theta/dt^2 = 4mg \times 3a\sin\theta$ $- \frac{3}{4}mg \times \frac{1}{2}a\sin\theta$ $- mg \times 3a\sin\theta$ $[= (69/8)\,mga\sin\theta]$	M1 A2	
	Approximate $\sin\theta$ by θ to show SHM: (M0 if wrong sign or $\cos\theta \approx \theta$ used)	$d^2\theta/dt^2 = -(69g/400a)\,\theta$ *or* $-(0{\cdot}1725\,g/a)\,\theta$	M1 A1	
	Find period T (AEF): (allow $g = 9{\cdot}8$ or $9{\cdot}81$)	$T = 2\pi/\sqrt{(69g/400a)}$ $= 40\pi\sqrt{(a/69g)}$ *or* $15{\cdot}1\sqrt{(a/g)}$ *or* $4{\cdot}78\sqrt{a}$	A1	[6]
6	Find prob. p of score of 6 on one throw:	$p = 5/36$ *or* $0{\cdot}139$	B1	
	Find mean of X:	$1/p = 36/5$ *or* $7{\cdot}2$	B1	[2]
	Formulate condition for N ($1 - q^N$ is M0):	$1 - q^{N-1} > 0{\cdot}95$	M1	
	Rearrange and take logs (any base) to give bound:	$(31/36)^{N-1} < 0{\cdot}05$ $N - 1 > \log 0{\cdot}05 / \log 31/36$	M1	
	Find N_{min}: ($N - 1 < 20{\cdot}03$ or $N - 1 = 20{\cdot}03$ earns M1 M1 A0)	$N - 1 > 20{\cdot}03$, $N_{min} = 22$	A1	[3]
7	Calculate sample mean:	$\bar{x} = 24{\cdot}6/9$ *or* $41/15$ *or* $2{\cdot}73[3]$	B1	
	Estimate population variance: (allow biased here: $0{\cdot}14$ *or* $0{\cdot}3742^2$)	$s^2 = (68{\cdot}5 - 24{\cdot}6^2/9)/8$ $= 63/400$ *or* $0{\cdot}1575$ *or* $0{\cdot}3969^2$	M1	
	State hypotheses (AEF; B0 for $\bar{x}$ ...):	H_0: $\mu = 2{\cdot}5$, H_1: $\mu > 2{\cdot}5$	B1	
	Calculate value of t (to 3 s.f.):	$t = (\bar{x} - 2{\cdot}5)/(s/\sqrt{9}) = 1{\cdot}76$	M1 A1	
	State or use correct tabular t-value (to 3 s.f.) : (or can compare $\bar{x}$ with $2{\cdot}5 + 0{\cdot}246 = 2{\cdot}746$)	$t_{8,\,0.95} = 1{\cdot}86[0]$	B1	
	State or imply valid method for conclusion e.g.:	Accept H_0 if $t <$ tabular value	M1	
	Conclusion (AEF, requires both values correct):	$1{\cdot}76\,[\pm 0{\cdot}02] < 1{\cdot}86$ so popln. mean not greater than $2{\cdot}5$	A1	[8]

Question	Mark Scheme Details		Part Mark	Total
8 **(i)**	Find or state distribution function $F(x)$ for $x \geqslant 0$: Use $F(0) = 0$ to find $F(x)$:	$F(x) = \int f(x)\,dx = -e^{-2x} + c$ $F(x) = 0\ (x < 0),\ 1 - e^{-2x}\ (x \geqslant 0)$	**M1** **A1**	[2]
(ii)	Find median value m from $F(m) = ½$:	$1 - e^{-2m} = ½$, $e^{2m} = 2$ $m = ½ \ln 2$ *or* 0·347	**M1** **M1 A1**	[3]
(iii)	Find or state $G(y)$ from $Y = e^X$ for $x \geqslant 0$: (allow < or ⩽ throughout) Find $g(y)$ by differentiation: State corresponding range of y:	$G(y) = P(Y < y) = P(e^X < y)$ $= P(X < \ln y) = F(\ln y)$ $= 1 - e^{-2 \ln y}$ [$= 1 - 1/y^2$] $g(y) = 2 / y^3$ for $y \geqslant 1$ [$g(y) = 0$ for $y < 1$]	 **M1 A1** **A1** **A1**	[4]
9	State (at least) null hypothesis in full: (AEF) Find exp. values using $150\ {}^6C_i\, q^{6-i}\, p^i$ with $p = 0·6$, $q = 0·4$ (to 3 s.f.): (allow A1 if only one error or if all values correct to 2 s.f.) Combine 0·6144 & 5·5296 since 1st exp. value < 5: Calculate χ^2 (result correct to 3 s.f.): State or use consistent tabular value (to 3 s.f.): [or if no or more cells combined: State or imply valid method for conclusion e.g.: Conclusion (AEF, requires both values correct): [data] (Allow A1 here for e.g. "It is a good fit") **S.C.** $150\ {}^6C_i\, q^i\, p^{6-i}$ can earn B1 M1 B1 M1 B1 M1 (max 6/10)	H_0: Given distribution fits data 0·6144 5·5296 20·736 41·472 46·656 27·9936 6·9984 O_i: 4 . . . E_i: 6·14[4] . . . $\chi^2 = 0·748 + 0·877 + 2·189$ $+ 1·606 + 0·144 + 3·570 = 9·13$ 6 cells: $\chi_{5,0.95}{}^2 = 11·07$ 7 cells: $\chi_{6,\,0.95}{}^2 = 12·59$ 5 cells: $\chi_{4,\,0.95}{}^2 = 9·488$ 4 cells: $\chi_{3,\,0.95}{}^2 = 7·815$ 3 cells: $\chi_{2,\,0.95}{}^2 = 5·991$] Accept H_0 if $\chi^2 <$ tabular value 9·13 [± 0·01] < 11·1 so distn. fits	**B1** **M1 A2** **B1** **M1 A1** **B1√** **M1** **A1**	[10]
10 **(i)**	Find $\sum x$ and $\sum y$ (M1 for either): **SC**: Allow M1 if 5 used instead of 6 (max 4/11), giving 62·97, 50·97 and $r = 0·961[5]$ Find correlation coefficient r, e.g.: 111·51	$(\sum x)^2 = 6\,(844·20 - 6 \times 36·66)$ $= 3775·44$ $\sum x = 61·2$ *or* $\bar{x} = 10·2$ $(\sum y)^2 = 6\,(481·5 - 6 \times 9·69)$ $= 2540·16$ $\sum y = 50·4$ *or* $\bar{y} = 8·4$ $S_{xy} = 625·59 - 61·2 \times 50·4/6 =$ $S_{xx} = 844·20 - 61·2^2/6$ *or* $6 \times 36·66 = 219·96$ $S_{yy} = 481·50 - 50·4^2/6$ *or* $6 \times 9·69 = 58·14$ $r = S_{xy} / \sqrt{(S_{xx} S_{yy})} = 0·986$	 **M1 A1** **A1** **M1 A1**	[5]

Question	Mark Scheme Details		Part Mark	Total
(ii)	Calculate gradients in both lines : *and*	$b_1 = S_{xy} / S_{xx} = 0{\cdot}507$ $b_2 = S_{xy} / S_{yy}$ *or* $r^2 / b_1 = 1{\cdot}918$	B1	
	Find one regression line, e.g. of y on x:	$y = 50{\cdot}4/6 + 0{\cdot}507\,(x - 61{\cdot}2/6)$ $= 0{\cdot}507\,x + 3{\cdot}23$	M1 A1	
	Find other regression line, e.g. of x on y:	$x = 61{\cdot}2/6 + 1{\cdot}918\,(y - 50{\cdot}4/6)$ $= 1{\cdot}92\,y - 5{\cdot}91$	A1	[4]
	SC: Use PA –1 if results only correct to 2 s.f.			
(iii)	Find x when $y = 6{\cdot}4$:	$x = 6{\cdot}36$ (allow 6·37 or 6·38)	B1	
	Valid comment on reliability, e.g.:	Reliable since $r \approx 1$ [√ on r] *or* 6·4 within range of y	B1	[2]
11A	Denoting angle between AB and CA produced by Φ and denoting mid-point of BC by E, say.			
	Take moments for rod about A (AEF):	$W \times a \sin \Phi + 2W \times (3a/2) \sin \Phi$ $= T \times AE$	M1 A1	
	Substituting $\Phi = 2\theta$ and $AE = 2a \sin \theta$:	$W \times a \sin 2\theta + 2W \times (3a/2) \sin 2\theta$ $= T \times 2a \sin \theta$	A1	
	Hence find tension T in terms of θ:	$T = 2W \sin 2\theta / \sin \theta$ $[= 4W \cos \theta]$	A1	
	Find T in terms of θ using Hooke's Law:	$T = 3W\,(BC - 3a/2) / (3a/2)$ $= W\,(8 \cos \theta - 3)$	M1 A1 A1	
	Equate to find $\cos \theta$:	$4 \cos \theta = 8 \cos \theta - 3$ $\cos \theta = ¾$ **A.G.**	M1 A1 B1	
	Hence find T (allow assumption of $\cos \theta = ¾$):	$T = 3W$	B1	[10]
	Find horizontal force X at A (ignore sign):	$X = T \sin \theta\ [= 3W\sqrt{7} / 4]$	B1	
	Find vertical force Y at A (ignore sign):	$Y = 3W - T \cos \theta\ [= 3W / 4]$		
	Find magnitude of force at A:	$\sqrt{(X^2 + Y^2)} = (3/\sqrt{2})\,W$ *or* $(3\sqrt{2} / 2)\,W$ *or* $2{\cdot}12\,W$	M1 A1	[4]

Question	Mark Scheme Details		Part Mark	Total
11B	Find Σx via sample mean $\bar{x}$:	$\Sigma x = 12\,\bar{x} = 12 \times ½\,(25{\cdot}17 + 26{\cdot}83)$	M1 A1	
		$= 12 \times 26 = 312$	M1	
	EITHER: Find estimated s.d. s_A, or s_A^2: with $t = t_{11,\,0.975} = 2{\cdot}201$	$t\,s_A / \sqrt{12} = ½\,(26{\cdot}83 - 25{\cdot}17)$		
	(to 3 s.f.)	$s_A = 0{\cdot}83\,\sqrt{12} / 2{\cdot}201 = 1{\cdot}306$ *or* $s_A^2 = 1{\cdot}706$	A1	
	Find Σx^2 from s_A or s_A^2: (M0 for $s_A^2 = \{\ldots\}/12$)	$s_A^2 = \{\Sigma x^2 - (\Sigma x)^2/12\} / 11$ $\Sigma x^2 = 11\,s_A^2 + (\Sigma x)^2/12 = 8130[{\cdot}8]$	M1 A1	[6]
	OR: Find estimated s.d. σ_A, or σ_A^2: with $t = t_{11,\,0.975} = 2{\cdot}201$	$t\,\sigma_A / \sqrt{11} = ½\,(26{\cdot}83 - 25{\cdot}17)$	(M1	
	(to 3 s.f.)	$\sigma_A = 0{\cdot}83\,\sqrt{11} / 2{\cdot}201 = 1{\cdot}25$ *or* $\sigma_A^2 = 1{\cdot}564$	A1	
	Find Σx^2 from σ_A or σ_A^2: (M0 for $\sigma_A^2 = \{\ldots\}/11$)	$\sigma_A^2 = \{\Sigma x^2 - (\Sigma x)^2/12\} / 12$ $\Sigma x^2 = 12\,\sigma_A^2 + (\Sigma x)^2/12 = 8130[{\cdot}8]$	M1 A1)	
	State hypotheses (B0 for $\bar{x}_A \ldots$), e.g.:	$H_0: \mu_A = \mu_B$, $H_1: \mu_A > \mu_B$	B1	
	Estimate *B*'s popln. variance (to 3 d.p..): (allow biased here: 4·578)	$s_B^2 = (4507{\cdot}62 - 177^2/7) / 6$ $[= 5{\cdot}341]$	M1	
	Find pooled estimate of common variance: (note s_B^2 may be implied, earning M1 above)	$s^2 = (11\,s_A^2 + 6\,s_B^2) / 17 = (18{\cdot}77 + 32{\cdot}05) / 17 = 2{\cdot}989$ *or* $1{\cdot}729^2$ (to 3 s.f.)	M1 A1	
	Calculate value of t (or $-t$): (to 3 s.f.)	$t = (26 - 177/7) / s\,\sqrt{(1/12 + 1/7)} = (26 - 25{\cdot}29) / s\,\sqrt{(1/12 + 1/7)} = 0{\cdot}7143 / 0{\cdot}8222 = 0{\cdot}869$	M1 A1	
	State or use correct tabular t value: (to 3 s.f.) (or can compare $\bar{x}_A - \bar{x}_B = 0{\cdot}7143$ with 1·536)	$t_{17,\,0.9} = 1{\cdot}333$	*B1	
	Consistent conclusion (AEF, √ on t, dep *B1):	$t <$ tabular value so Petra's belief not supported *or* wing span of *A* not greater	B1√	[8]