FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

- Candidates need to be careful to eliminate arithmetic and numerical errors in their answers.
- Candidates need to read the questions carefully and answer as required.
- Candidates need to avoid sign errors.

General comments

The scripts for this paper were generally of a good quality. There were a number of high quality scripts and many showing evidence of sound learning. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a sound standard. The work on proof by mathematical induction showed improvement and the vector question was done well by the vast majority of candidates.

A very high proportion of scripts had substantial attempts at all eleven questions. There were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the vector and proof by mathematical induction work, already mentioned, there was good work on questions involving calculus, matrices and rational functions. The knowledge of the polar coordinates section of the syllabus continued to remain rather insecure.

Comments on specific questions

Question 1

This question produced very variable responses. Some candidates were able to find the correct cartesian equation and then could not produce the correct sketch. Others were able to produce a correct sketch, but not obtain the correct cartesian equation. Overall, there were few completely correct solutions.

Answers: $x + y = 2$, $(2,0)$.

Question 2

In both parts of this question, most candidates knew the appropriate method. There were occasional errors with arithmetic, usually in the second part, involving division by fractions.

Answers: **(i)** $\frac{8}{3}$, **(ii)** $\frac{3}{2}$.

Question 3

This question was done extremely competently by the vast majority of candidates. Only a small number were unable to obtain the correct complementary function. Most candidates were able to carry out the necessary algebra to obtain the particular integral. There were some candidates who had some other variable where they should have had t, in either the complementary function or particular integral. In cases where they recovered from such errors, they suffered no penalty.

Answer: $x = e^{-2t}(A\cos 3t + B\sin 3t) + 2t^2 - t + 1$.

Question 4

Most candidates were able to establish the first result correctly. The word 'hence' implied that they were expected to use it, along with the method of differences, to obtain the second result. Most candidates were able to use the given answer for the second part, in order to obtain the sum of the squares formula. Only the most able candidates were able to make progress in the final part. Few saw that it was necessary to add the sum of the squares of even numbers up to $(n-1)^2$ to the sum of the squares up to n^2. Among those who saw this initial step, it was rare to find cases where there was progress in finding the sum of the squares of even numbers up to $(n-1)^2$.

Answer: $\frac{1}{2}n^2(n+1)$.

Question 5

The majority of candidates were successful in using integration by parts in order to find the given reduction formula. There were a far greater number of candidates than usual, who were able to perform each step in the proof by mathematical induction, which followed. A small number did not realise that in order to prove the base case, they needed to find I_0. Instead they found I_1. Relatively few candidates lost the final mark for not stating an adequate conclusion, which needed to refer to the result being true 'for all positive integers', not just 'for all n' or 'for $n \geq 1$'.

Question 6

The first result was obtained by a large number of candidates, who used de Moivre's theorem in the intended manner. There were a small number who used de Moivre's theorem in a rather obscure manner, but were, nevertheless, successful. This alternative was allowed. Those who obtained the result, without using de Moivre's theorem received no credit for this part of the question. The second part of the question defeated all but the most able candidates. Most candidates made good progress with the final two parts of the question, making use of the given results for the first two parts. Many candidates realised that the final part relied on knowing about the sum of the roots of a polynomial equation.

Question 7

A considerable number of candidates were able to score full marks on this question on rational functions. A rather larger number of candidates did not continue their division process far enough to obtain the correct oblique asymptote, in the first part of the question. This resulted in them thinking that the oblique asymptote was $y = \lambda x$. Consequently, they were only able to score one mark for their sketch, although all three marks could be obtained for the middle part of the question. Most candidates adopted a successful strategy for moving from a correct form of the gradient function to the required result, in this middle part of the question. Sketches in the final part were, mostly well drawn, with only a few candidates being penalised for poor forms at infinity, or not showing the required intersections with the axes.

Answers: $x = 2$, $y = \lambda x + 1$, (0,0), (3,0).

Question 8

The majority of candidates knew the appropriate techniques for both parts of this question. Many were able to do both parts correctly. Weaker candidates found difficulty in showing $\frac{ds}{dt} = t^2 + \frac{1}{t}$. Some of those, who made this step successfully, made errors with fractional indices later in the question. Another common error was to take the lower limit of integration as 0, rather than 1.

Answers: $\frac{26}{3} + \ln 3$ or 9.77; $\pi\left(\frac{160\sqrt{3}}{3} - \frac{64}{27}\right)$ or 283.

Question 9

Most candidates were able to find the normal vector to the plane Π and then use a scalar product to show that the line l was parallel to Π, in the first part of the question. In the second part, the most direct way was to find the value of the parameter for the second line at the point where the line intersected Π. The cartesian equation of Π was more useful in doing this. Some candidates successfully solved a system of linear equations. In the final part there were a good number of correct solutions. A common error was to use the scalar product of $\begin{pmatrix} 6 \\ 5 \\ -10 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \\ -8 \end{pmatrix}$, where the vector product was required. Other methods involving the use of the scalar product coupled with Pythagoras were successful.

Answers: $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$; $\sqrt{8}$ or $2\sqrt{2}$ or 2.83.

Question 10

Most candidates could correctly write down the eigenvalues, by inspection, but a small number found the characteristic equation and solved it. The fact that there was only one mark for this part of the question should have alerted them to the fact that there was a simple answer. Most were then able to make a good attempt, with varying degrees of success, at finding the corresponding eigenvectors. The marks for the matrices **P** and **D** were awarded on a follow through basis, so most candidates were able to score these. Some candidates needed to exercise a bit more care in finding the inverse of the matrix **P**. There were numerous sign errors and from time to time the transpose of the required matrix appeared. The calculation of $\mathbf{A}^n$ produced mixed results. Those who made slight errors were still able to get the final mark for the limit, following through from their answer for $\mathbf{A}^n$.

Answers: 1, 2, 3; $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$; $\mathbf{P} = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$;

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A}^n = \begin{pmatrix} 1 & [-4 + 4.2^n] & [14 - 12.2^n - 2.3^n] \\ 0 & 2^n & [-3.2^n + 3^{n+1}] \\ 0 & 0 & 3^n \end{pmatrix};$$

$$3^{-n}\mathrm{A}^n \to \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \text{ as } n \to \infty.$$

Question 11

EITHER

The initial proof was successfully negotiated by the stronger candidates. There were various incorrect assumptions made in an attempt to produce a correct argument by the weaker candidates. The latter could continue with the remainder of the question, using the given result, often quite effectively. The careful candidates were able to get some and, in many cases, all of the required numerical results. Errors, at any stage, unfortunately, did incur loss of further accuracy marks. Only the stronger candidates saw the result $\sum \alpha^2\beta^3 = S_2S_3 - S_5$, which was necessary to do the final part.

Answers: **(i)** $S_2 = 6$, $S_4 = 26$; **(ii)** $S_3 = -15$, $S_5 = -75$, -15.

OR

Candidates attempting this, less popular, alternative were generally successful in reducing the given matrix to echelon form and deducing the dimension of R, the range space. They were then able to write down two linearly independent three-dimensional vectors as a basis for R in part **(ii)**. There were few correct attempts to find the cartesian equation of R. Few candidates appreciated that they were looking for a plane. There were a good number of candidates who successfully found a basis for the null space of T, the linear transformation. Even though they had no cartesian equation for R in part **(ii)**, a number of candidates were able to solve a set of equations and find the value of k correctly. Only the best candidates were able to find the general solution of the given equation. A number of candidates showed a fundamental misunderstanding of the problem. Method marks were earned by those making a good attempt at a particular solution and then adding it to a linear combination of the vectors forming the basis of the nullspace.

Answers: **(ii)** $\left\{\begin{pmatrix}2\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\4\\2\end{pmatrix}\right\}$, $2x - y + z = 0$; **(iii)** $\left\{\begin{pmatrix}-3\\2\\0\\1\end{pmatrix}, \begin{pmatrix}2\\-3\\1\\0\end{pmatrix}\right\}$;

$k = -9$, $\mathbf{x} = \begin{pmatrix}5\\-2\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\2\\0\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\1\\0\end{pmatrix}$.

FURTHER MATHEMATICS

Paper 9231/12

Paper 12

Key messages

- Candidates need to be careful to eliminate arithmetic and numerical errors in their answers.
- Candidates need to read the questions carefully and answer as required.
- Candidates need to avoid sign errors.

General comments

The scripts for this paper were generally of a good quality. There were a number of high quality scripts and many showing evidence of sound learning. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a sound standard. The work on proof by mathematical induction showed improvement and the vector question was done well by the vast majority of candidates.

A very high proportion of scripts had substantial attempts at all eleven questions. There were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the vector and proof by mathematical induction work, already mentioned, there was good work on questions involving calculus, matrices and rational functions. The knowledge of the polar coordinates section of the syllabus continued to remain rather insecure.

Comments on specific questions

Question 1

This question produced very variable responses. Some candidates were able to find the correct cartesian equation and then could not produce the correct sketch. Others were able to produce a correct sketch, but not obtain the correct cartesian equation. Overall, there were few completely correct solutions.

Answers: $x + y = 2$, (2,0).

Question 2

In both parts of this question, most candidates knew the appropriate method. There were occasional errors with arithmetic, usually in the second part, involving division by fractions.

Answers: **(i)** $\frac{8}{3}$, **(ii)** $\frac{3}{2}$.

Question 3

This question was done extremely competently by the vast majority of candidates. Only a small number were unable to obtain the correct complementary function. Most candidates were able to carry out the necessary algebra to obtain the particular integral. There were some candidates who had some other variable where they should have had t, in either the complementary function or particular integral. In cases where they recovered from such errors, they suffered no penalty.

Answer: $x = e^{-2t}(A\cos 3t + B\sin 3t) + 2t^2 - t + 1$.

Question 4

Most candidates were able to establish the first result correctly. The word 'hence' implied that they were expected to use it, along with the method of differences, to obtain the second result. Most candidates were able to use the given answer for the second part, in order to obtain the sum of the squares formula. Only the most able candidates were able to make progress in the final part. Few saw that it was necessary to add the sum of the squares of even numbers up to $(n - 1)^2$ to the sum of the squares up to n^2. Among those who saw this initial step, it was rare to find cases where there was progress in finding the sum of the squares of even numbers up to $(n - 1)^2$.

Answer: $\frac{1}{2}n^2(n+1)$.

Question 5

The majority of candidates were successful in using integration by parts in order to find the given reduction formula. There were a far greater number of candidates than usual, who were able to perform each step in the proof by mathematical induction, which followed. A small number did not realise that in order to prove the base case, they needed to find I_0. Instead they found I_1. Relatively few candidates lost the final mark for not stating an adequate conclusion, which needed to refer to the result being true 'for all positive integers', not just 'for all n' or 'for $n \geq 1$'.

Question 6

The first result was obtained by a large number of candidates, who used de Moivre's theorem in the intended manner. There were a small number who used de Moivre's theorem in a rather obscure manner, but were, nevertheless, successful. This alternative was allowed. Those who obtained the result, without using de Moivre's theorem received no credit for this part of the question. The second part of the question defeated all but the most able candidates. Most candidates made good progress with the final two parts of the question, making use of the given results for the first two parts. Many candidates realised that the final part relied on knowing about the sum of the roots of a polynomial equation.

Question 7

A considerable number of candidates were able to score full marks on this question on rational functions. A rather larger number of candidates did not continue their division process far enough to obtain the correct oblique asymptote, in the first part of the question. This resulted in them thinking that the oblique asymptote was $y = \lambda x$. Consequently, they were only able to score one mark for their sketch, although all three marks could be obtained for the middle part of the question. Most candidates adopted a successful strategy for moving from a correct form of the gradient function to the required result, in this middle part of the question. Sketches in the final part were, mostly well drawn, with only a few candidates being penalised for poor forms at infinity, or not showing the required intersections with the axes.

Answers: $x = 2$, $y = \lambda x + 1$, (0,0), (3,0).

Question 8

The majority of candidates knew the appropriate techniques for both parts of this question. Many were able to do both parts correctly. Weaker candidates found difficulty in showing $\frac{\mathrm{d}s}{\mathrm{d}t} = t^2 + \frac{1}{t}$. Some of those, who made this step successfully, made errors with fractional indices later in the question. Another common error was to take the lower limit of integration as 0, rather than 1.

Answers: $\frac{26}{3} + \ln 3$ or 9.77; $\pi\left(\frac{160\sqrt{3}}{3} - \frac{64}{27}\right)$ or 283.

Question 9

Most candidates were able to find the normal vector to the plane Π and then use a scalar product to show that the line l was parallel to Π, in the first part of the question. In the second part, the most direct way was to find the value of the parameter for the second line at the point where the line intersected Π. The cartesian equation of Π was more useful in doing this. Some candidates successfully solved a system of linear equations. In the final part there were a good number of correct solutions. A common error was to use the scalar product of $\begin{pmatrix} 6 \\ 5 \\ -10 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \\ -8 \end{pmatrix}$, where the vector product was required. Other methods involving the use of the scalar product coupled with Pythagoras were successful.

Answers: $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$; $\sqrt{8}$ or $2\sqrt{2}$ or 2.83.

Question 10

Most candidates could correctly write down the eigenvalues, by inspection, but a small number found the characteristic equation and solved it. The fact that there was only one mark for this part of the question should have alerted them to the fact that there was a simple answer. Most were then able to make a good attempt, with varying degrees of success, at finding the corresponding eigenvectors. The marks for the matrices **P** and **D** were awarded on a follow through basis, so most candidates were able to score these. Some candidates needed to exercise a bit more care in finding the inverse of the matrix **P**. There were numerous sign errors and from time to time the transpose of the required matrix appeared. The calculation of $\mathbf{A}^n$ produced mixed results. Those who made slight errors were still able to get the final mark for the limit, following through from their answer for $\mathbf{A}^n$.

Answers: 1, 2, 3; $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$; $\mathbf{P} = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$;

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A}^n = \begin{pmatrix} 1 & [-4 + 4.2^n] & [14 - 12.2^n - 2.3^n] \\ 0 & 2^n & [-3.2^n + 3^{n+1}] \\ 0 & 0 & 3^n \end{pmatrix};$$

$$3^{-n}\mathrm{A}^n \to \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \text{ as } n \to \infty.$$

Question 11

EITHER

The initial proof was successfully negotiated by the stronger candidates. There were various incorrect assumptions made in an attempt to produce a correct argument by the weaker candidates. The latter could continue with the remainder of the question, using the given result, often quite effectively. The careful candidates were able to get some and, in many cases, all of the required numerical results. Errors, at any stage, unfortunately, did incur loss of further accuracy marks. Only the stronger candidates saw the result $\sum \alpha^2\beta^3 = S_2S_3 - S_5$, which was necessary to do the final part.

Answers: **(i)** $S_2 = 6$, $S_4 = 26$; **(ii)** $S_3 = -15$, $S_5 = -75$, -15.

OR

Candidates attempting this, less popular, alternative were generally successful in reducing the given matrix to echelon form and deducing the dimension of *R*, the range space. They were then able to write down two linearly independent three-dimensional vectors as a basis for *R* in part **(ii)**. There were few correct attempts to find the cartesian equation of *R*. Few candidates appreciated that they were looking for a plane. There were a good number of candidates who successfully found a basis for the null space of *T*, the linear transformation. Even though they had no cartesian equation for *R* in part **(ii)**, a number of candidates were able to solve a set of equations and find the value of *k* correctly. Only the best candidates were able to find the general solution of the given equation. A number of candidates showed a fundamental misunderstanding of the problem. Method marks were earned by those making a good attempt at a particular solution and then adding it to a linear combination of the vectors forming the basis of the nullspace.

Answers: **(ii)** $\left\{\begin{pmatrix}2\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\4\\2\end{pmatrix}\right\}$, $2x - y + z = 0$; **(iii)** $\left\{\begin{pmatrix}-3\\2\\0\\1\end{pmatrix}, \begin{pmatrix}2\\-3\\1\\0\end{pmatrix}\right\}$;

$k = -9$, $\mathbf{x} = \begin{pmatrix}5\\-2\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\2\\0\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\1\\0\end{pmatrix}$.

FURTHER MATHEMATICS

Paper 9231/13

Paper 13

Key messages

- Candidates need to read the questions carefully and answer as required.
- Candidates need to be careful to eliminate arithmetic and numerical errors in their answers.
- Candidates need to avoid sign errors.

General comments

The standard of work was generally of a high quality and some candidates made excellent responses to the questions. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard. Calculus work, in particular, was very strong.

All of the scripts had substantial attempts at all twelve questions. There were no misreads and no rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the calculus work, already mentioned, candidates tackled the questions on series, roots of equations and vectors confidently. This group of candidates were able to cope well with the questions on complex numbers, proof by induction and polar curves, which are often done less well.

Comments on specific questions

Question 1

These candidates readily saw that $\sum_{n+1}^{2n} = \sum_{1}^{2n} - \sum_{1}^{n}$ and were able to perform the necessary algebra to obtain the given result.

Question 2

Several candidates attempted a solution using row operations and, finding progress difficult, resorted to equating the appropriate determinant to zero. All using this method were able to complete the question satisfactorily.

Answer: All real values except $a = \frac{1}{2}$ and $a = -2$.

Question 3

All of the candidates were familiar with proof by mathematical induction and were able to complete this straightforward example of it without undue difficulty.

Question 4

All candidates knew how to use the vector product to find the area of the triangle *ABC*. Occasionally the final mark was lost for not finding the perpendicular height correctly.

Answers: **(i)** $\frac{1}{2}\sqrt{3}$, **(ii)** $\sqrt{\frac{3}{14}}$.

Question 5

Most candidates were able to sketch the polar curve correctly. They were also familiar with the area of a sector formula and were able to complete the necessary integration.

Answer: $\left[\pi + \frac{5}{2}\sqrt{3}\right]$.

Question 6

Most candidates were able to find $\left(\frac{\mathrm{ds}}{\mathrm{dt}}\right)$ in its simplified form. The surface area formula was well known and most candidates were able to complete the integration, in order to obtain the surface area generated by revolution.

Answer: 24π.

Question 7

Most candidates were able to find the value of $\sum \alpha\beta$ from the given information. Finding the value of $\alpha\beta\gamma$ proved difficult for some, but they were able to solve the given cubic equation and obtain the final two marks.

Answers: $\sum \alpha\beta = 1$, $x = -1, 2, 3$.

Question 8

There was much competent work in evidence in this question. Most candidates were able to show the initial result and make a substantial attempt to combine the binomial theorem with de Moivre's theorem in order to find the sum of the given series.

Answer: $2^n \cos^n \frac{1}{2}\theta \sin \frac{n}{2}\theta$.

Question 9

Work on this question was of a good standard. The occasional candidate thought that the oblique asymptote was $y = x$, because the division process was not continued far enough. The second part was most readily done by rearranging the equation of the curve into a standard form quadratic equation in x, so that the discriminant could be used to establish the printed result. Some candidates delayed this part until they had found the turning points of C, but some neglected, when adopting this approach, to show whether the turning point was a local maximum or a local minimum. For those with the correct asymptotes, the sketch was usually correct.

Answers: $x = 2$, $y = x - 1$, $(1,-1)$, $(3,3)$.

Question 10

Candidates were mostly able to differentiate the implicit equation correctly and obtain the correct relationship between x and y when $\frac{dy}{dx} = 0$. The coordinates of the turning point of C were usually found correctly, apart from the occasional error with fractional indices. The nature of the turning point was usually determined using the second derivative, but other methods were used which were perfectly acceptable.

Answers: $y = x^2$, $\left(2^{\frac{1}{3}}, 2^{\frac{2}{3}}\right)$, maximum.

Question 11

The initial differentiation caused little difficulty and candidates could see how it was relevant to the integration by parts, which was necessary to establish the reduction formula. The integration by substitution was generally done well, with only the occasional error with signs. Those who were unable to correctly obtain the reduction formula were able to apply it in the final part of the question, in order to find I_4.

Answer: $\frac{\pi}{32}$.

Question 12

EITHER

This was the less popular of the two alternatives for the final question. Those doing it were able to complete the initial proof and find the corresponding eigenvalues and eigenvector. They then knew that it was necessary to find the eigenvalues of the matrix **B**, and thus those of the matrix **C**, using the initial result. At this stage some minor inaccuracies had occurred and the entries in the matrices **P** and **D** were sometimes incorrect, particularly if the eigenvalues of the matrix **C** were not squared, when writing down the matrix **D**.

Answers: 0, 1 ; $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$; $\mathbf{P} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$.

OR

Those doing this alternative were able to make a good attempt at finding both the complementary function and the particular integral, which were invariably both correct, leading to the correct general solution. Occasionally the calculation of one of the arbitrary constants produced a small error. The occasional error or omission also crept in to the final part where the answer was given and so full working had to be shown.

Answers: $x = e^{-3t}(A\cos 2t + B\sin 2t) + 3\cos 2t + 4\sin 2t$;
$x = e^{-3t}(2\cos 2t - \sin 2t) + 3\cos 2t + 4\sin 2t$.

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found; candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

General comments

Most candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a very strong preference for the Statistics option, though the small minority of candidates who chose the Mechanics option frequently produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1**, **6**, **7** and **8**. **Question 2** was found to be challenging by many candidates.

Advice to candidates to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 10**, and the directions of motion of particles, as in **Question 4**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 8** and **9** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true, rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, some rounded intermediate results to this same accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Almost all candidates derived the given moment of inertia of the body correctly, usually by finding the moments of inertia of the two discs and the rod about their centres, applying the parallel axes theorem to each and finally summing the three moments of inertia about *O*. An occasionally seen and equally acceptable alternative is to treat the two parts *AO* and *OB* of the rod separately. Some candidates attempted the question by simply writing down a sum of six or more terms without explanation; though acceptable if all the terms are correct, any error risks losing substantial credit if the reason for including the various terms is thereby rendered unclear.

Question 2

This was found to be the most challenging question on the paper, with only a minority of candidates using a valid approach to the first part. This is best answered by first equating the couple on the disc due to the tension T of the string to the product of the moment of inertia of the disc and its angular acceleration. A second equation for T comes from applying Newton's second law of motion to the particle, which is descending in a straight line and is subject to a net downward force of $1{\cdot}5\,g - T$, enabling T to be eliminated and the angular acceleration found. Some candidates instead treated the system as if the particle is attached to the right-hand rim of the disc at a point level with the centre, effectively increasing the moment of inertia in the first equation by $1{\cdot}5 \times 0{\cdot}4^2$. Taking the couple in the first equation to be due solely to the weight of the particle then yields immediately the correct value of the (instantaneous) angular acceleration, but none of the candidates using this approach attempted to justify it or even adequately explain what they were doing. A very common but wholly invalid approach is to take $T = 1{\cdot}5\,g$, effectively ignoring the motion of the particle, and giving an incorrect value for the angular acceleration of 30 rad s^{-2}. The second part is most easily solved by noting that the square of the angular velocity ω is twice the product of the angular acceleration, found earlier, and the given angle turned. Multiplication of ω by the radius 0·4 then gives the required speed of the particle. Alternatively conservation of energy may be used, but in this case the change in potential energy of the particle must be equated to the change in kinetic energy of both the disc and the particle.

Answers: **(i)** 13·6 rad s^{-2}; **(ii)** 1·51 m s^{-1}.

Question 3

Most candidates found the given tension successfully by combining conservation of energy, which gives an expression for the speed v when AP makes an angle θ with the downward vertical, with an application of Newton's second law of motion to the particle P. It is unnecessary, though not wrong, to find the speed of P at the lowest point before using this to find v. The key to the second part is that the tension should not become zero prior to P reaching the highest point, so substitution of $\theta = \pi$ in the expression for T produces the required least possible value of x/a. Some candidates instead required that v should not become zero, but this is insufficient here since the particle is not constrained to move in a circle.

Answer: 3/5.

Question 4

The component of P's speed parallel to the barrier is unchanged in the first collision, while the component normal to the barrier is changed in magnitude by a factor ⅓. Combining these results gives a speed v after the collision of $u/\sqrt{3}$, and thus the kinetic energy is reduced by $\frac{2}{3}$ as required. A minor error by some candidates was to take P's mass as m rather than $2\,m$ when discussing its kinetic energy, though the fraction lost is unaltered by this mistake. More serious faults were to simply and wrongly take $v = \frac{1}{3}u$ without considering the components of P's speed, or to assume without any explicit justification that P's motion after striking the barrier makes an angle 30° with it. Many candidates were able to show that P's speed after colliding with Q is $\frac{1}{3}u$ by equating the change in P's momentum to the given impulse. The same process for Q, or the equivalent conservation of momentum equation, should yield Q's speed after the collision but sign errors were common. These also occurred frequently when applying Newton's law of restitution to find the coefficient of restitution between P and Q, which the question specifies should be given here in an exact form.

Answer: **(ii)** $\sqrt{3} - 1$.

Question 5

While most candidates equated the two opposing tensions in order to show that $AP = 4a$ when P is in equilibrium, some introduced an unknown length other than AP without defining it or indicating it on a diagram. Similar uncertainty sometimes occurred when applying Newton's second law to the motion of the particle after being released. Ideally this equation should concern the displacement x of P from the equilibrium point since this produces the standard form of the SHM equation, and if another distance is introduced then candidates should be careful to define it and preferably relate it to x. Once the SHM equation has been obtained in the form $d^2x/dt^2 = -8gx/a$ the period T follows from $T = 2\pi/\omega$ since ω^2 is here $8g/a$. The maximum speed is found from another standard SHM formula ωa, and the result should as usual be given in its simplest form. A few candidates considered motion perpendicular to the line AB, underlining the need to read questions sufficiently carefully.

Answer: **(ii)** $\sqrt{(8ag)}$.

Question 6

The first two probabilities were usually found correctly from $q^{19}p$ and $1 - q^4$ respectively where $p = 0{\cdot}2$ and $q = 1 - p = 0{\cdot}8$, and candidates should give these and other inexact numerical results to 3 significant figures as required by the instructions for the paper. The final part is a little more challenging, requiring the least value of the integer n which satisfies $1 - q^n \geq 0{\cdot}95$, and some of those candidates who instead solved the corresponding equality for n to give 13·4 mistakenly chose 13 December as the required date.

Answers: **(i)** 0·00288; **(ii)** 0·590; **(iii)** 14 December.

Question 7

The given form of the distribution function G(y) was usually verified by first integrating f(x) to give F(x) and then noting that $G(y) = F(y^{1/3})$. The median value m of Y usually presented few problems either, being found from $G(m) = ½$. Finally E(Y) may be found by integrating yg(y) or equivalently x^3f(x) with appropriate limits in either case, though some candidates mistakenly integrated the wrong function such as yG(y).

Answers: **(i)** 24·8; **(ii)** 27·3.

Question 8

Finding both the equation of the regression line and the product moment correlation coefficient r for the sample is straightforward, with the relevant expressions given in the *List of Formulae*. Apart from an occasional arithmetical error most candidates experienced little difficulty, but they should retain additional significant figures in their intermediate working in order to ensure 3 significant figure accuracy in their results. The test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho > 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the magnitude of the previously calculated value of r with the tabular value 0·621 leads to a conclusion of there being evidence of positive correlation.

Answers: **(i)** $y = 5{\cdot}38 + 0{\cdot}483x$; **(ii)** 0·671.

Question 9

The required 95% confidence interval was found correctly by most candidates, using the unbiased estimate 1·737 of the population variance and a tabular t-value of 2·447. The most suitable assumptions in the second part are that the distribution of the lengths is also normal for area B, and that its population variance is the same as that for area A's distribution. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. The unbiased estimate 1·536 of the common population variance may be used to calculate a t-value of 1·10. Since it is a one-tail test, comparison with the tabulated value of 1·812 leads to acceptance of the null hypothesis, namely that the mean length for area A is not greater than that for area B.

Answers: [6·98, 9·42].

Question 10

EITHER

This was much less popular than the Statistics alternative discussed below, being attempted by fewer than ten per cent of the candidates. Many of those who chose this optional question did, however, make good attempts. In order to verify the given lower bounds for the two coefficients of friction it is necessary to find the ratio of the friction and normal reaction at one of the points where a sphere rests on the plane and at one of the points at which two spheres are in contact. There is of course symmetry about the vertical line through the centre of C. These ratios may be found from a variety of resolution and moment equations, all equally valid. However, in order to minimise the time and effort in answering questions such as this, it is advisable to choose those moment and resolution equations which will lead most quickly to the desired result. Thus taking moments for the sphere A (or B) about its point of contact P with the ground yields a bound on μ' immediately. By symmetry and vertical resolution of forces for the system the normal reaction at P is $3W/2$, and taking moments for the sphere A about its point of contact with C shows that the frictional force at P (or indeed at any of the points of contact) is $(W \sin \theta) / 2(1 + \cos \theta)$. Combining these two results leads to the given lower bound on μ. Candidates are advised to define any symbols they introduce for forces, either in writing or by use of a diagram in their answer, since otherwise equations containing unexplained symbols such as F, N, R may be so unclear as to not gain credit.

OR

Finding E(X) by integrating xf(x) over [2, 4] rarely presented problems, but candidates were often confused over how to show that that the resulting value 2·8 is within 10% of the estimated mean 2·69. This requires only that 2·8 lies in the interval $2{\cdot}69 \pm 0{\cdot}269$, but not that 2·69 lies in the interval $2{\cdot}8 \pm 0{\cdot}28$. Other candidates mistakenly attempted to find a confidence interval on the basis of a normal distribution. Most were able to show how the expected frequency for the specified interval is obtained, namely by integrating 60f(x) over this interval. The goodness of fit test was also often carried out well, with the last two cells being combined because the final expected frequency is less than 5. Comparison of the calculated value 3·78 of χ^2 with the critical value 6·251 leads to acceptance of the null hypothesis, namely that the given probability density function f is appropriate.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found; candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

General comments

Most candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a very strong preference for the Statistics option, though the small minority of candidates who chose the Mechanics option frequently produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1**, **6**, **7** and **8**. **Question 2** was found to be challenging by many candidates.

Advice to candidates to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 10**, and the directions of motion of particles, as in **Question 4**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 8** and **9** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true, rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, some rounded intermediate results to this same accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Almost all candidates derived the given moment of inertia of the body correctly, usually by finding the moments of inertia of the two discs and the rod about their centres, applying the parallel axes theorem to each and finally summing the three moments of inertia about *O*. An occasionally seen and equally acceptable alternative is to treat the two parts *AO* and *OB* of the rod separately. Some candidates attempted the question by simply writing down a sum of six or more terms without explanation; though acceptable if all the terms are correct, any error risks losing substantial credit if the reason for including the various terms is thereby rendered unclear.

Question 2

This was found to be the most challenging question on the paper, with only a minority of candidates using a valid approach to the first part. This is best answered by first equating the couple on the disc due to the tension T of the string to the product of the moment of inertia of the disc and its angular acceleration. A second equation for T comes from applying Newton's second law of motion to the particle, which is descending in a straight line and is subject to a net downward force of $1{\cdot}5\,g - T$, enabling T to be eliminated and the angular acceleration found. Some candidates instead treated the system as if the particle is attached to the right-hand rim of the disc at a point level with the centre, effectively increasing the moment of inertia in the first equation by $1{\cdot}5 \times 0{\cdot}4^2$. Taking the couple in the first equation to be due solely to the weight of the particle then yields immediately the correct value of the (instantaneous) angular acceleration, but none of the candidates using this approach attempted to justify it or even adequately explain what they were doing. A very common but wholly invalid approach is to take $T = 1{\cdot}5\,g$, effectively ignoring the motion of the particle, and giving an incorrect value for the angular acceleration of 30 rad s^{-2}. The second part is most easily solved by noting that the square of the angular velocity ω is twice the product of the angular acceleration, found earlier, and the given angle turned. Multiplication of ω by the radius $0{\cdot}4$ then gives the required speed of the particle. Alternatively conservation of energy may be used, but in this case the change in potential energy of the particle must be equated to the change in kinetic energy of both the disc and the particle.

Answers: **(i)** 13·6 rad s^{-2}; **(ii)** 1·51 m s^{-1}.

Question 3

Most candidates found the given tension successfully by combining conservation of energy, which gives an expression for the speed v when AP makes an angle θ with the downward vertical, with an application of Newton's second law of motion to the particle P. It is unnecessary, though not wrong, to find the speed of P at the lowest point before using this to find v. The key to the second part is that the tension should not become zero prior to P reaching the highest point, so substitution of $\theta = \pi$ in the expression for T produces the required least possible value of x/a. Some candidates instead required that v should not become zero, but this is insufficient here since the particle is not constrained to move in a circle.

Answer: 3/5.

Question 4

The component of P's speed parallel to the barrier is unchanged in the first collision, while the component normal to the barrier is changed in magnitude by a factor ⅓. Combining these results gives a speed v after the collision of $u/\sqrt{3}$, and thus the kinetic energy is reduced by $\frac{2}{3}$ as required. A minor error by some candidates was to take P's mass as m rather than $2\,m$ when discussing its kinetic energy, though the fraction lost is unaltered by this mistake. More serious faults were to simply and wrongly take $v = \frac{1}{3}u$ without considering the components of P's speed, or to assume without any explicit justification that P's motion after striking the barrier makes an angle 30° with it. Many candidates were able to show that P's speed after colliding with Q is $\frac{1}{3}u$ by equating the change in P's momentum to the given impulse. The same process for Q, or the equivalent conservation of momentum equation, should yield Q's speed after the collision but sign errors were common. These also occurred frequently when applying Newton's law of restitution to find the coefficient of restitution between P and Q, which the question specifies should be given here in an exact form.

Answer: **(ii)** $\sqrt{3} - 1$.

Question 5

While most candidates equated the two opposing tensions in order to show that $AP = 4a$ when P is in equilibrium, some introduced an unknown length other than AP without defining it or indicating it on a diagram. Similar uncertainty sometimes occurred when applying Newton's second law to the motion of the particle after being released. Ideally this equation should concern the displacement x of P from the equilibrium point since this produces the standard form of the SHM equation, and if another distance is introduced then candidates should be careful to define it and preferably relate it to x. Once the SHM equation has been obtained in the form $d^2x/dt^2 = -8gx/a$ the period T follows from $T = 2\pi/\omega$ since ω^2 is here $8g/a$. The maximum speed is found from another standard SHM formula ωa, and the result should as usual be given in its simplest form. A few candidates considered motion perpendicular to the line AB, underlining the need to read questions sufficiently carefully.

Answer: **(ii)** $\sqrt{(8ag)}$.

Question 6

The first two probabilities were usually found correctly from $q^{19}p$ and $1 - q^4$ respectively where $p = 0{\cdot}2$ and $q = 1 - p = 0{\cdot}8$, and candidates should give these and other inexact numerical results to 3 significant figures as required by the instructions for the paper. The final part is a little more challenging, requiring the least value of the integer n which satisfies $1 - q^n \geq 0{\cdot}95$, and some of those candidates who instead solved the corresponding equality for n to give 13·4 mistakenly chose 13 December as the required date.

Answers: **(i)** 0·00288; **(ii)** 0·590; **(iii)** 14 December.

Question 7

The given form of the distribution function G(y) was usually verified by first integrating f(x) to give F(x) and then noting that $G(y) = F(y^{1/3})$. The median value m of Y usually presented few problems either, being found from G(m) = ½. Finally E(Y) may be found by integrating yg(y) or equivalently $x^3f(x)$ with appropriate limits in either case, though some candidates mistakenly integrated the wrong function such as yG(y).

Answers: **(i)** 24·8; **(ii)** 27·3.

Question 8

Finding both the equation of the regression line and the product moment correlation coefficient r for the sample is straightforward, with the relevant expressions given in the *List of Formulae*. Apart from an occasional arithmetical error most candidates experienced little difficulty, but they should retain additional significant figures in their intermediate working in order to ensure 3 significant figure accuracy in their results. The test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho > 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the magnitude of the previously calculated value of r with the tabular value 0·621 leads to a conclusion of there being evidence of positive correlation.

Answers: **(i)** $y = 5{\cdot}38 + 0{\cdot}483x$; **(ii)** 0·671.

Question 9

The required 95% confidence interval was found correctly by most candidates, using the unbiased estimate 1·737 of the population variance and a tabular t-value of 2·447. The most suitable assumptions in the second part are that the distribution of the lengths is also normal for area B, and that its population variance is the same as that for area A's distribution. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. The unbiased estimate 1·536 of the common population variance may be used to calculate a t-value of 1·10. Since it is a one-tail test, comparison with the tabulated value of 1·812 leads to acceptance of the null hypothesis, namely that the mean length for area A is not greater than that for area B.

Answers: [6·98, 9·42].

Question 10

EITHER

This was much less popular than the Statistics alternative discussed below, being attempted by fewer than ten per cent of the candidates. Many of those who chose this optional question did, however, make good attempts. In order to verify the given lower bounds for the two coefficients of friction it is necessary to find the ratio of the friction and normal reaction at one of the points where a sphere rests on the plane and at one of the points at which two spheres are in contact. There is of course symmetry about the vertical line through the centre of *C*. These ratios may be found from a variety of resolution and moment equations, all equally valid. However, in order to minimise the time and effort in answering questions such as this, it is advisable to choose those moment and resolution equations which will lead most quickly to the desired result. Thus taking moments for the sphere *A* (or *B*) about its point of contact *P* with the ground yields a bound on μ' immediately. By symmetry and vertical resolution of forces for the system the normal reaction at *P* is $3W/2$, and taking moments for the sphere *A* about its point of contact with *C* shows that the frictional force at *P* (or indeed at any of the points of contact) is $(W \sin \theta) / 2(1 + \cos \theta)$. Combining these two results leads to the given lower bound on μ. Candidates are advised to define any symbols they introduce for forces, either in writing or by use of a diagram in their answer, since otherwise equations containing unexplained symbols such as *F*, *N*, *R* may be so unclear as to not gain credit.

OR

Finding E(X) by integrating xf(x) over [2, 4] rarely presented problems, but candidates were often confused over how to show that that the resulting value 2·8 is within 10% of the estimated mean 2·69. This requires only that 2·8 lies in the interval $2{\cdot}69 \pm 0{\cdot}269$, but not that 2·69 lies in the interval $2{\cdot}8 \pm 0{\cdot}28$. Other candidates mistakenly attempted to find a confidence interval on the basis of a normal distribution. Most were able to show how the expected frequency for the specified interval is obtained, namely by integrating 60f(x) over this interval. The goodness of fit test was also often carried out well, with the last two cells being combined because the final expected frequency is less than 5. Comparison of the calculated value 3·78 of χ^2 with the critical value 6·251 leads to acceptance of the null hypothesis, namely that the given probability density function *f* is appropriate.

FURTHER MATHEMATICS

Paper 9231/23

Paper 23

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

General comments

Candidates attempted all the questions, and most performed very well. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option, though candidates who chose the Mechanics option produced good attempts.

Comments on specific questions

Question 1

The possible values of k are found by equating the magnitudes of the radial acceleration $(k - t^2)^2/1{\cdot}5$ and transverse acceleration $2t$ with $t = 3$, and solving the resulting quadratic. Finding the angular velocity and acceleration is an unnecessary complication.

Answers: 6, 12.

Question 2

Derivation of the given expression for the magnitude of the reaction R requires that the net radial force acting on the particle as a result of R, $4\,mg$ and $mg\cos\theta$ be equated to mv^2/a, followed by substitution for the speed v using conservation of energy relative to the initial position. Some candidates wrongly thought that the required inequality in the second part depends on R, but this is not the case here since the bead is threaded on the circular wire. Instead the bead will reach the highest point provided $v \ge 0$ when $\theta = \pi$, or equivalently if the initial kinetic energy $\tfrac{1}{2}mkga$ is no less than the change $2\,mga$ in potential energy.

Answer: $k \ge 4$.

Question 3

The majority of candidates answered this question well, though some made only limited progress through including a non-zero frictional force at C. The given reaction at C is easily verified by taking moments for BC about B. Part **(ii)** was solved most often by taking moments for AB about B, though some candidates used the equally valid alternative of taking moments for the system about A. The least possible value of μ follows from the ratio of the friction and normal reaction at A, found by horizontal and vertical resolution of the forces on the system.

Answer: $\tfrac{1}{4}\sqrt{3}$.

Question 4

Both part **(i)** and part **(ii)** are solved by using conservation of momentum and Newton's law of restitution, and these presented no difficulty to most candidates. The third part was found more challenging by some, due in some cases to confusion over the final directions of the three particles. Once it is shown that *A* reverses direction and *B* ultimately moves in the same direction as *A*, it only remains to show that the speed of *A* exceeds that of *B* so that they cannot again collide. *C* is of course irrelevant after its collision with *B* since these two particles must then be moving in opposite directions.

Answer: **(ii)** $u(1 + e)(1 - 2e)/9$.

Question 5

The required moment of inertia was usually found without difficulty using the parallel axes theorem, either for each rod individually or less often after finding the moment of inertia of the frame about its centre of mass. Applying conservation of energy produced an expression in a variety of forms for the required angular velocity, usually performed correctly apart from infrequent arithmetical errors. When finding the value of k using $\theta = 90°$, not all candidates realised that the radius of rotation of the point *C* is $2\sqrt{2}a$.

Answers: $\sqrt{\{(3g/5a)(2 - \sqrt{2}(1 - \cos\theta))\}}$; 1·68.

Question 6

The distribution function F(x) of X is readily found by integration. After stating or finding the value 6 of the mean of X, the required probability is simply F(6) – ½. Many candidates found the value of the median, but this is not necessary.

Answers: **(i)** $1 - e^{-x/6}$ $(x \geq 0)$, 0 otherwise; **(ii)** 0·132.

Question 7

The necessary assumption of a normal distribution was usually stated, and candidates knew how to find the distribution function though the correct t-tabular value was not always used. A complete answer to part **(ii)** requires consideration of all three variables in the width of the confidence interval, all of which tend to reduce the width calculated from the combined 15 results.

Answer: **(i)** [105, 117].

Question 8

Most candidates made a good attempt at this question, first using a value ¾ in the binomial distribution to find the expected values, and then usually combining appropriate values because some are less than 5. Comparison of the calculated value 1·50 of χ^2 with the critical value 2·706 leads to acceptance of the null hypothesis, namely that the binomial distribution is appropriate.

Question 9

The hypotheses were usually stated correctly and the test conducted appropriately to conclude that at the 10% significance level the mean breaking strengths are not the same. Candidates should note that if they choose to use a test which assumes equal variances for the distributions of *P* and *Q* then this assumption should be stated explicitly. The more usual approach is not to make such an assumption and instead estimate the population variance 0·04023 of the combined distribution, leading to a comparison of the magnitude of the calculated z-value 1·82 with the critical value 1·645.

Question 10

This question was very well answered, with candidates obtaining full or nearly full marks. Since the rubric for the paper requires that non-exact numerical answers should be given correct to 3 significant figures, candidates should use at least 4 in their intermediate working in order to produce the required accuracy in the final result. Since there is no obvious reason for preferring one regression line over the other, either may be used to estimate the time taken by a delegate who travelled 100 km.

Answers: $y = -1{\cdot}45 + 0{\cdot}967x$, $x = 12{\cdot}1 + 0{\cdot}888y$; 101 or 105 minutes; 0·926.

Question 11

EITHER

Few candidates chose this option, but many made excellent attempts. Care should be taken to produce the correct negative sign in the standard equation of simple harmonic motion.

Answer: $9a/16$.

OR

The constant k was invariably found correctly by equating the area under the graph to 1, and the equations of the two line segments and hence f(x) then follow immediately. Most candidates appreciated that the probably density function g(y) of Y is obtained by finding in turn F(x), G(y) and finally g(y), though not all were able to carry out this process correctly. E(Y) may be found by integrating yg(y) or equivalently x^2f(x) with appropriate limits in either case, and finally the given relation between the medians is readily obtained.

Answer: **(i)** 1/16 $(0 \le y \le 4)$, $1/8\sqrt{y}$ $(4 < y \le 25)$, 0 otherwise.

