FURTHER MATHEMATICS

Paper 9231/11

Key Messages:

- Candidates need to make sure that they communicate their working in full, particularly in questions where the answer is given
- Candidates need to read the questions very carefully so that they don't miss key words such as 'hence' and 'exact' and do complete the questions fully
- Candidates should be self-critical when answers start to extend over many lines or even pages, and look back for simplifications, errors or alternative methods that they can implement
- Candidates need to label all their sketch graphs fully

General Comments:

The paper was accessible to the majority of candidates, who seemed to have sufficient time to attempt all questions. The strongest candidates completed both alternatives of question eleven and even weaker candidates were able to complete the paper and re-try earlier questions at the end. There were some appropriate challenges within the paper which tested depth of understanding, and allowed the strongest candidates to show their insight into the underlying mathematics. Most candidates presented their work well, and showed clear working and the number of misreads and careless mistakes was pleasingly small. Algebraic handling was generally sound, and most candidates appreciated when exact answers were required. Some marks were lost when candidates did not appreciate the significance of the wording of the question. The standard of graph sketching was variable.

Question 1:

Most candidates tackled this correctly remembering how to find the second derivative for a parametric function using the chain rule, though a few found $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. Some candidates substituted multiple angle formulae for the power expressions, and this caused difficulties in the second stage. Others did not simplify $\frac{\mathrm{d}y}{\mathrm{d}x}$ and ended up with complicated quotient rule calculations, and some then showed great stamina in reaching a correct answer eventually. A four mark question does not require several pages of working, so candidates would have been well advised to check back and look for the earlier simplifications.

Question 2:

This question was mainly well done, with most students remembering the correct form of the complementary function when the auxiliary equation has repeated roots. It is important that students practise all variations of the auxiliary equation to build confidence. Most students realised that they needed a three term quadratic form for the particular integral and did not fall into the trap of mirroring the two terms on the right hand side of the equation. Some candidates started with a cubic and successfully found the quadratic. A number of candidates lost marks through failing to use the correct variables, using x instead of t. Again this is an area where practice with different variables reinforces learning.

Answer: $x = Ae^{-2t} + Bte^{-2t} - \frac{1}{2}t^2 + t + 1$

Question 3:

It was good to see some very clear arguments where candidates had clearly learned how to set out their inductive proofs, and their good understanding helped them carry out all the steps correctly. Sometimes the words defining the overall structure of an induction proof were incorrect. Most candidates knew to differentiate to show the inductive step to find H_{k+1} from H_k . A few became confused with difference methods and a small number became confused between the constants and variables, but the majority differentiated correctly.

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The first part of this question caused relatively few problems apart from a small number of candidates who found the 30th term, instead of the sum of the first 30 terms. In the second part, some candidates lost the square root sign in the denominator but most candidates squared the two sides of the inequality correctly, though this was one of the few sources of careless algebraic slips in the paper.
Some candidates solved the quadratic inequality in the second part correctly, but failed to relate their values of *n* to the actual problem, where *n* must be an integer. It is a useful checking mechanism to cross reference final answers against the context of the question.

Answer: **(i)** 4.820 **(ii)** $n = 55$

Question 5:

Apart from very occasional sign problems, the first part of this question proved straightforward. In the second part, stronger candidates appreciated the significance of the roots being real and indeed the question clearly asked for 'the value' (singular) of *r*. Other candidates were less careful in their reading and gave two solutions without checking that both led to real roots.

Answer: **(i)** $p = -15$, $q = 71$ **(ii)** $\alpha = 3$, $\alpha \neq 12$, $r = -105$

Question 6:

The first two parts of this question were well done, with the majority of candidates knowing a method for finding the eigenvectors. Most found the third eigenvalue directly, though a few took the slightly more time-consuming route of using the characteristic equation to find all the eigenvalues. Having found **P**, it was good to see most candidates knew how to work out its inverse correctly with few errors in this process. There was some extra work put into the last part of the question by candidates who thought that they had to calculate **D** by multiplying out the right hand side of the given equation.

Answer: **(i)** $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ oe **(ii)** $\lambda = -2$

(iii) (for example) **D** = $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, **P** = $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $\mathbf{P}^{-1}$ = $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & 2 \end{pmatrix}$ and multiples/permutations.

Question 7:

Most candidates were able to perform the row reduction to find the rank of **M**, and stronger candidates went on to find the basis for the null space, though a few were unable to complete this question. More able candidates were able to use this basis to find the required values of *a* and *b* where others took the more laborious route of multiplying out.

Answer: **(i)** $r(\mathbf{M}) = 4 - 2 = 2$, basis (for example) $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$ **(ii)** $\alpha = 50$, $b = 30$

Question 8:

Most candidates differentiated and correctly checked the discriminant of the resulting (quadratic) derivative. However some candidates tried to find whether the function was bounded. Disappointingly a number of candidates failed to complete the division necessary to find the oblique asymptote and mistakenly gave the oblique asymptote as $y = 2x$ having left a top heavy fraction after division. The vertical asymptote was usually identified successfully, and the roots found correctly. Sketches were mainly well done, though some allowed their graph to curve away from the asymptotes, not towards them.

Answer: **(ii)** Vertical asymptote $x = -1$, Oblique asymptote $y = 2x + 2$

Question 9:

This proved to be the question which caused most difficulty, with weaker candidates struggling to prove the given reduction formula. There were some excellent attempts from stronger candidates who spotted connections between the integrals and a wide variety of correct methods were seen. It was pleasing to find that most candidates remembered how to find the mean value of a function, and many were able to use the given identity to complete the second part, though some seemed to forget to substitute I_3 back into their formula for the final accuracy mark.

Answer: Mean Value = $\dfrac{I^3}{e-1} = \dfrac{6-2e}{e-1}$

Question 10:

The first part of this question posed few problems, though a few candidates caused difficulties for themselves by substituting to form expressions for $\sin 5\theta$ in terms only of $\sin\theta$ and similarly for $\cos 5\theta$. Others omitted to show the final step of dividing by $\cos^5\theta$ which is essential when working towards a given answer. The second part of the question proved difficult to all but the strongest mathematicians, who appreciated the need to show the five roots of $\tan 5\theta = 0$ and explain how these reduced to the given roots. Candidates who remembered the appropriate trigonometric identity connecting the tangent and secant correctly picked up the final three marks without difficulty.

Answer: **(iii)** $y^2 - 12y + 16 = 0$

Question 11 EITHER:

This was a less popular choice than the alternative, but neatly done by most of those who chose it. Most candidates were able to find the appropriate perpendicular vectors using the cross product, and find equations for lines and planes competently. Some found the intersection of the line *AP* with the plane *OBC*, others showed more insight into the geometry.

Answer: **(ii)** $\mathbf{q} = \dfrac{1}{3}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

Question 11 OR:

Some sketches showed no scaling at all, perhaps copied from calculator screens, but most were well drawn and tables of value were included. Most candidates were able to find the area enclosed by the two half lines, remembering the formula and making the appropriate trigonometric substitution. The last part of the question caused more difficulties. Some candidates knew how to find the arc length, s, but did not relate this to the derivative, $\frac{\mathrm{d}s}{\mathrm{d}\theta}$. Others found the derivative after finding the arc length i.e. by differentiation. The trigonometrical manipulation was good, apart from a small number of sign errors. Unfortunately, some strong candidates found the expression for $\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2$ but then forgot to complete the question by using it to calculate the arc length required.

Answer: **(ii)** $a^2\left(\frac{3}{4}\pi + 2\right)$ **(iii)** $4\sqrt{2a}$

FURTHER MATHEMATICS

Paper 9231/12

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- Candidates should be self-critical when answers start to extend over many lines or even pages, and look back for simplifications, errors or alternative methods that they can implement
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General Comments:

The paper was accessible to the majority of candidates, who seemed to have sufficient time to attempt all questions. The strongest candidates completed both alternatives of question eleven and even weaker candidates were able to complete the paper and re-try earlier questions at the end. There were some appropriate challenges within the paper which tested depth of understanding, and allowed the strongest candidates to show their insight into the underlying mathematics. Most candidates presented their work well, and showed clear working and the number of misreads and careless mistakes was pleasingly small. Algebraic handling was generally sound, and most candidates appreciated when exact answers were required. Some marks were lost when candidates did not appreciate the significance of the wording of the question. The standard of graph sketching was variable.

Question 1:

Most candidates tackled this correctly remembering how to find the second derivative for a parametric function using the chain rule, though a few found $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. Some candidates substituted multiple angle formulae for the power expressions, and this caused difficulties in the second stage. Others did not simplify $\frac{\mathrm{d}y}{\mathrm{d}x}$ and ended up with complicated quotient rule calculations, and some then showed great stamina in reaching a correct answer eventually. A four mark question does not require several pages of working, so candidates would have been well advised to check back and look for the earlier simplifications.

Question 2:

This question was mainly well done, with most students remembering the correct form of the complementary function when the auxiliary equation has repeated roots. It is important that students practise all variations of the auxiliary equation to build confidence. Most students realised that they needed a three term quadratic form for the particular integral and did not fall into the trap of mirroring the two terms on the right hand side of the equation. Some candidates started with a cubic and successfully found the quadratic. A number of candidates lost marks through failing to use the correct variables, using x instead of t. Again this is an area where practice with different variables reinforces learning.

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Some candidates solved the quadratic inequality in the second part correctly, but failed to relate their values of *n* to the actual problem, where *n* must be an integer. It is a useful checking mechanism to cross reference final answers against the context of the question.

Answer: **(i)** 4.820 **(ii)** $n = 55$

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Most candidates were able to perform the row reduction to find the rank of **M**, and stronger candidates went on to find the basis for the null space, though a few were unable to complete this question. More able candidates were able to use this basis to find the required values of *a* and *b* where others took the more laborious route of multiplying out.

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Most candidates differentiated and correctly checked the discriminant of the resulting (quadratic) derivative. However some candidates tried to find whether the function was bounded. Disappointingly a number of candidates failed to complete the division necessary to find the oblique asymptote and mistakenly gave the oblique asymptote as $y = 2x$ having left a top heavy fraction after division. The vertical asymptote was usually identified successfully, and the roots found correctly. Sketches were mainly well done, though some allowed their graph to curve away from the asymptotes, not towards them.

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FURTHER MATHEMATICS

Paper 9231/13

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Answer: **(ii)** $a^2\left(\frac{3}{4}\pi + 2\right)$ **(iii)** $4\sqrt{2a}$

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key messages

To score full marks in the paper, candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers. Annotating a diagram in the question paper is insufficient, since this will not be seen by the Examiners.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Questions 1, 4, 7 and 10** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 1** and the directions of motion of particles, as in **Question 2**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 9 and 10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Since the tension T in the rope and the normal reactions R_A, R_B at A, B must be found, it follows that three independent equations must be formulated. Most candidates chose to resolve forces horizontally and vertically and to take moments about a point such as A, B, C or D, though some used more than one such moment equation. A rarely-seen alternative point is the intersection of the lines of action of the two normal reactions, which has the merit of yielding T immediately. However finding the moment of T seemed to cause the most difficulty, which suggests that moments about C or D might be the most straightforward choice.

Answer: $T = \frac{1}{2}W/\sin\alpha$; $R_A = 3W/2$; $R_B = \frac{1}{2}W/\tan\alpha$.

Question 2

Almost all candidates were able to formulate and then solve two equations for the speeds v_A and v_B of the spheres A and B after the first collision by means of conservation of momentum and Newton's restitution equation. A second application of the restitution principle gives B's speed v_B' after its collision with the wall as $0{\cdot}4\,v_B$ and equating this to v_A yields the required value of e. Even though the direction of motion of B is opposite to that of A after colliding with the wall, it is necessary to equate their speeds and not their velocities, since the introduction of an inappropriate minus sign gives of course an incorrect value of e. The final part can be approached in different ways, and so candidates are advised to give some indication of what they are doing, in case an arithmetic error leads to an incorrect answer for the required distance x. Thus, for example, the time $(d - x)/v_A$ taken by A may be equated to the total time $d/v_B + x/v_B'$ taken by B. Alternatively the distance $0{\cdot}4d$ moved by A when B reaches the wall may be found from $(d/v_B)v_A$, and x must be one-half the remaining distance between A and the wall since it is given that A and B then move with the same speed before colliding. Candidates should note that applying conservation of momentum and Newton's restitution equation once again to find the speeds of A and B after their second collision is of no assistance in answering the question, and consequently earns no credit.

Answer: $\frac{1}{3}(2 - e)\,u$, $\frac{2}{3}(1 + e)\,u$; $\frac{2}{3}$; $0{\cdot}3\,d$.

Question 3

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Answer: $1{\cdot}2$; $2\pi\sqrt{(3a/8g)}$; $0{\cdot}49$.

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Answer: $4\sqrt{(ag/5)}$; $(5\ mg/4)\ (6/5 - 3 \cos \theta)$; 0·4.

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The general method for finding the required 95% confidence interval was known to most candidates, though the correct unbiased estimate 103/225 ≈ 0·4578 of the population variance and the tabular t-value of 2·262 were not always used.

Answer: (21·8, 22·8).

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Answers: **(i)** 10·2; **(ii)** 341/35 or 9·74.

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Most candidates chose to use the regression line of y on x with x = 72 in order to estimate correctly the corresponding value of y, though x on y can also be used, giving an alternative estimate of 62. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho > 0$, though some wrongly stated them in terms of r which conventionally relates of course to the sample and not the population. The value 0·797 of the product moment correlation coefficient r was usually found correctly from the standard formula, and comparison with the tabular value 0·707 leads to a conclusion of there being evidence of non-zero correlation.

Answer: 58.

Question 10 (Mechanics)

This optional question was attempted by only a small minority of the candidates, who mostly experienced little difficulty in verifying the two given moments of inertia. These require the use of standard formulae and the parallel axes theorem where necessary to formulate and then sum the individual moments of inertia of the lamina and the rod. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. The reference in the final part of the question to the periods of small oscillations rightly suggested to most candidates attempting it that they should derive an SHM equation in each of the two cases. This is readily done by equating the product of the moment of inertia and the angular acceleration to the couple acting on the object when displaced by a small angle θ. Approximating $\sin\theta$ by θ then yields the standard form of the SHM equation and hence the period $T = 2\pi/\omega$ for each of the two cases. Finally these are verified to be equal when $m = M$. Candidates should be aware that the constant parameter ω in the SHM equation is not equal to $d\theta/dt$ even though the same symbol is sometimes used to denote the latter, and thus finding $d\theta/dt$ from conservation of energy at some fixed angular displacement and equating it to the SHM parameter ω is wholly invalid. As always, candidates should read the question carefully, noting in particular in the final part that expressions involving m and M should be found for each of the two periods, and only then setting $m = M$ rather than taking m and M to be equal from the start.

Answer: $2\pi\sqrt{\{2(8m + M)a / (3m + M)g\}}$; $2\pi\sqrt{\{(17m + M)a / 4\,mg\}}$.

Question 10 (Statistics)

An appropriate additional assumption when comparing the Royal and Majestic plants is that their populations share a common variance. Although an explicit statement of this assumption was often omitted, the majority of the many candidates attempting this optional question did indeed attempt to find a pooled estimate 0·431 of this common variance, and hence a calculated t-value of 2·13. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Since it is a two-tailed test, comparison with the tabulated value of 2·086 leads to acceptance of the alternative hypothesis, namely that the population mean masses are not the same. The second part proved more challenging, possibly because many candidates were unsure how to estimate an appropriate variance. Since nothing is known about the Crown plants apart from their mean, the estimate of the population variance obtained from the 10 Royal plants should be used, giving a calculated t-value of 2. Comparison with the corresponding tabular value of 1·833 again leads to acceptance of the alternative hypothesis, namely here that the population mean mass is greater than 3·8, so Farmer *A*’s claim is justified.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

To score full marks in the paper, candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

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Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers. Annotating a diagram in the question paper is insufficient, since this will not be seen by the Examiners.

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Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 1** and the directions of motion of particles, as in **Question 2**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 9 and 10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Since the tension T in the rope and the normal reactions R_A, R_B at A, B must be found, it follows that three independent equations must be formulated. Most candidates chose to resolve forces horizontally and vertically and to take moments about a point such as A, B, C or D, though some used more than one such moment equation. A rarely-seen alternative point is the intersection of the lines of action of the two normal reactions, which has the merit of yielding T immediately. However finding the moment of T seemed to cause the most difficulty, which suggests that moments about C or D might be the most straightforward choice.

Answer: $T = \frac{1}{2}W / \sin\alpha$; $R_A = 3W/2$; $R_B = \frac{1}{2}W / \tan\alpha$.

Question 2

Almost all candidates were able to formulate and then solve two equations for the speeds v_A and v_B of the spheres A and B after the first collision by means of conservation of momentum and Newton's restitution equation. A second application of the restitution principle gives B's speed v_B' after its collision with the wall as $0{\cdot}4\, v_B$ and equating this to v_A yields the required value of e. Even though the direction of motion of B is opposite to that of A after colliding with the wall, it is necessary to equate their speeds and not their velocities, since the introduction of an inappropriate minus sign gives of course an incorrect value of e. The final part can be approached in different ways, and so candidates are advised to give some indication of what they are doing, in case an arithmetic error leads to an incorrect answer for the required distance x. Thus, for example, the time $(d - x)/v_A$ taken by A may be equated to the total time $d/v_B + x/v_B'$ taken by B. Alternatively the distance $0{\cdot}4d$ moved by A when B reaches the wall may be found from $(d/v_B)v_A$, and x must be one-half the remaining distance between A and the wall since it is given that A and B then move with the same speed before colliding. Candidates should note that applying conservation of momentum and Newton's restitution equation once again to find the speeds of A and B after their second collision is of no assistance in answering the question, and consequently earns no credit.

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FURTHER MATHEMATICS

Paper 9231/23
Paper 23

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Answer: 58.

Question 10 (Mechanics)

This optional question was attempted by only a small minority of the candidates, who mostly experienced little difficulty in verifying the two given moments of inertia. These require the use of standard formulae and the parallel axes theorem where necessary to formulate and then sum the individual moments of inertia of the lamina and the rod. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. The reference in the final part of the question to the periods of small oscillations rightly suggested to most candidates attempting it that they should derive an SHM equation in each of the two cases. This is readily done by equating the product of the moment of inertia and the angular acceleration to the couple acting on the object when displaced by a small angle θ. Approximating $\sin\theta$ by θ then yields the standard form of the SHM equation and hence the period $T = 2\pi/\omega$ for each of the two cases. Finally these are verified to be equal when $m = M$. Candidates should be aware that the constant parameter ω in the SHM equation is not equal to $d\theta/dt$ even though the same symbol is sometimes used to denote the latter, and thus finding $d\theta/dt$ from conservation of energy at some fixed angular displacement and equating it to the SHM parameter ω is wholly invalid. As always, candidates should read the question carefully, noting in particular in the final part that expressions involving m and M should be found for each of the two periods, and only then setting $m = M$ rather than taking m and M to be equal from the start.

Answer: $2\pi\sqrt{\{2(8m + M)a / (3m + M)g\}}$; $2\pi\sqrt{\{(17m + M)a / 4mg\}}$.

Question 10 (Statistics)

An appropriate additional assumption when comparing the Royal and Majestic plants is that their populations share a common variance. Although an explicit statement of this assumption was often omitted, the majority of the many candidates attempting this optional question did indeed attempt to find a pooled estimate 0·431 of this common variance, and hence a calculated t-value of 2·13. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Since it is a two-tailed test, comparison with the tabulated value of 2·086 leads to acceptance of the alternative hypothesis, namely that the population mean masses are not the same. The second part proved more challenging, possibly because many candidates were unsure how to estimate an appropriate variance. Since nothing is known about the Crown plants apart from their mean, the estimate of the population variance obtained from the 10 Royal plants should be used, giving a calculated t-value of 2. Comparison with the corresponding tabular value of 1·833 again leads to acceptance of the alternative hypothesis, namely here that the population mean mass is greater than 3·8, so Farmer A's claim is justified.