CAMBRIDGE INTERNATIONAL EXAMINATIONS

Cambridge International Advanced Level

MARK SCHEME for the October/November 2015 series

9231 FURTHER MATHEMATICS

9231/23 Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2015 series for most Cambridge IGCSE®, Cambridge International A and AS Level components and some Cambridge O Level components.

CAMBRIDGE
International Examinations

Question Number	Mark Scheme Details			Part Mark	Total
1	Find 3 independent equations for T, R_A, R_B :				
	Resolve horizontally:	$R_B = T \cos \alpha$	M1 A1		
	Resolve vertically:	$R_A = W + T \sin \alpha$	M1 A1		
	Take moments about A: (a may be omitted from moment eqns)	$R_B\, 3a \sin \theta = W(3a/2) \cos \theta$ $+ T a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ *or* $+ T a \sin (\alpha + \theta)$ *or* $+ T\, 3a \cos \theta \sin \alpha$	M1 A1		
	Take moments about B:	$R_A 3a \cos \theta = W(3a/2) \cos \theta$ $+ T\, 2a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ *or* $+ T\, 2a \sin (\alpha + \theta)$ *or* $+ T\, 3a \sin \theta \cos \alpha$	(M1 A1)		
	Take moments about C:	$R_A\, a \cos \theta + W(a/2) \cos \theta$ $= R_B\, 2a \sin \theta$	(M1 A1)		
	Take moments about D:	$R_A\, 3a \cos \theta - W(3a/2) \cos \theta$ $= R_B\, 3a \sin \theta$	(M1 A1)		
	Solve for T, R_A, R_B (AEF in W and α):	$T = W / 2 \sin \alpha$ *or* $\tfrac{1}{2}W \operatorname{cosec} \alpha$ $R_A = 3W / 2$ $R_B = W / 2 \tan \alpha$ *or* $\tfrac{1}{2}W \cot \alpha$	B1 B1 B1	9	9
2	For A & B use conservation of momentum,	e.g.: $2mv_A + mv_B = 2mu$ (allow $2v_A + v_B = 2u$)	M1		
	Use Newton's law of restitution (consistent signs):	$v_B - v_A = eu$	M1		
	Combine to find v_A and v_B :	$v_A = (2 - e)\, u/3$, $v_B = 2(1 + e)\, u/3$	A1, A1	4	
	Find e from $v_A = \lvert v_B' \rvert$ with $v_B' = [-]\, 0{\cdot}4\, v_B$:	$(2 - e) = 0{\cdot}8\,(1 + e)$, $e = 2/3$	M1 A1	2	
	EITHER: Equate times in terms of reqd. distance x: [speeds need not be found: Use $v_A = v_B' = 0{\cdot}4\, v_B$ to solve for x:	$(d - x)/ v_A = d/v_B + x/v_B'$ (AEF) $v_A = v_B' = 4u/9$, $v_B = 10u/9$] $d - x = 0{\cdot}4\, d + x$, $x = 0{\cdot}3\, d$	M1 A1 M1 A1		
	OR: Find dist. moved by A when B reaches wall:	$d_A = (d/v_B)\, v_A = 0{\cdot}4\, d$	(M1 A1)		
	Find reqd. distance x:	$x = \tfrac{1}{2}(d - d_A) = 0{\cdot}3\, d$	(M1 A1)	4	10

Question Number	Mark Scheme Details			Part Mark	Total
3	Find k by equating equilibrium tensions: (vertical motion can earn M1 only)	$mg\,(a/2)/a = 2mg\,(3a/2 - ka)\,/\,ka$ $½ = 3/k - 2, \quad k = 6/5$ *or* $1{\cdot}2$	**M1 A1** **A1**	3	
	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m\,\mathrm{d}^2x/\mathrm{d}t^2 = -mg\,(a/2 + x)/a$ $+ 2mg\,(3a/2 - ka - x)\,/\,ka$ *or* $m\,\mathrm{d}^2y/\mathrm{d}t^2 = +mg\,(a/2 - y)/a$ $- 2mg\,(3a/2 - ka + y)\,/\,ka$	**M1 A2**		
	Simplify to give standard SHM eqn, e.g.: **S.R.**: B1 if no derivation (max 2/5)	$\mathrm{d}^2x/\mathrm{d}t^2 = -(1 + 2/k)gx\,/\,a$ $= -8gx/3a$	**A1**		
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$: ($\sqrt{}$ on ω)	$T = 2\pi\sqrt{(3a/8g)}$ *or* $\pi\sqrt{(3a/2g)}$ *or* $3{\cdot}85\sqrt{(a/g)}$ *or* $1{\cdot}22\sqrt{a}$ [s]	**B1**$\sqrt{}$	5	
	Substitute values in $v^2 = \omega^2\,(x_0^{\,2} - x^2)$:	$0{\cdot}7^2 = (8g/3a)\{(0{\cdot}2a)^2 - (0{\cdot}05a)^2\}$	**M1 A1**		
	Solve to find numerical value of a:	$0{\cdot}49 = (8g/3) \times 0{\cdot}0375a, \; a = 0{\cdot}49$	**A1**	3	**11**

Question Number	Mark Scheme Details			Part Mark	Total
4	*EITHER:* Find tension at top from $F = ma$ vertically:	$T = mu^2/a - mg$	B1		
	OR: Use energy at e.g. θ to upward vertical:	$\frac{1}{2} mv^2 = \frac{1}{2} mu^2 + mga(1 - \cos\theta)$			
	Find tension T by using $F = ma$ radially:	$T' = mv^2/a - mg\cos\theta$			
	Eliminate v^2:	$= mu^2/a + mg(2 - 3\cos\theta)$			
	Find T at top by taking $\theta = 0$:	$T = mu^2/a - mg$	(B1)		
	Find u_{min} by requiring $T \geqslant 0$ at top [or $T > 0$]:	$u^2/a - g \geqslant 0$ so $u_{min} = \sqrt{ag}$ **A.G.**	B1	2	
	Find v at bottom from conservation of energy:	$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$	M1		
		$v^2 = ag + 4ag,\ v = \sqrt{(5ag)}$	A1		
	Find new speed V from conservation of momentum:	$m'V = mv$ with $m' = m + \frac{1}{4}m$	M1		
		$V = 4v/5 = 4\sqrt{(ag/5)}$ *or* $(4/5)\sqrt{(5ag)}$ AEF	A1	4	
	Find w^2 at angle θ from conservation of energy: (condone m instead of m' here since cancels out)	$\frac{1}{2} m'w^2 = \frac{1}{2} m'V^2 - m'ga(1 + \cos\theta)$	M1 A1		
		$[\ w^2 = ag(6/5 - 2\cos\theta)]$			
	S.R. Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3\cos\theta)$]	$\frac{1}{2} m'w^2 = \frac{1}{2} mu^2 + mga(1 - \cos\theta) - \frac{1}{4}mga(1 + \cos\theta)$	(B1)		
	Find tension T' there by using $F = ma$ radially:	$T' = m'w^2/a - m'g\cos\theta$	B1		
	Eliminate w^2:	$= m'V^2/a - m'g(2 + 3\cos\theta)$	A1		
	Substitute for m' and V: AEF	$= (5mg/4)(6/5 - 3\cos\theta)$ *or* $3mg/2 - (15/4)mg\cos\theta$	A1	5	
	Find $\cos\theta$ when string becomes slack from $T' = 0$: **S.R.** Allow if found from $T' = mg(6/5 - 3\cos\theta)$	$\cos\theta = \frac{1}{3} \times 6/5 = 2/5$ *or* $0{\cdot}4$	M1 A1	2	**13**
5	Find or use sample mean	$\bar{x} = 222{\cdot}8 / 10 = 22{\cdot}28$			
	<u>and</u> estimate population variance: (allow biased here: $0{\cdot}412$ *or* $0{\cdot}642^2$)	$s^2 = 4{\cdot}12 / 9 = 0{\cdot}458$ *or* $103/225$ *or* $0{\cdot}677^2$	M1		
	Find confidence interval (allow z in place of t) e.g.:	$22{\cdot}28 \pm t\sqrt{(0{\cdot}458 / 10)}$	M1 A1		
	Use of correct tabular value:	$t_{9,\ 0.975} = 2{\cdot}26[2]$	A1		
	Evaluate C.I. correct to 3 s.f.:	$22{\cdot}3 \pm 0{\cdot}48[4]$ *or* $[21{\cdot}8, 22{\cdot}8]$	A1	5	**5**

Question Number	Mark Scheme Details			Part Mark	Total
6	Find prob. p of head from mean = 2 × variance:	$1/p = 2 \times (1 - p)/p^2$, $p = ⅔$ **A.G.**	**M1 A1**	2	
(i)	Find $P(X = 4)$ (denoting $1 - p$ by q [= ⅓]):	$P(X = 4) = q^3 \times p$ $= 2/81$ *or* $0·0247$	**B1**	1	
(ii)	Find or state $P(X > 4)$:	$P(X > 4)$ [$= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)$] $= q^4$ $= 1/81$ *or* $0·0123$	**M1 A1**	2	
(iii)	Formulate condition for N: Take logs (any base) to give bound for N: Find N_{min}: ($N < 6·29$ *or* $N = 6·29$ earns M2 A0)	$1 - q^N > 0·999$, [$(⅓)^N < 0·001$] $N >$ log 0·001 / log ⅓ $N > 6·29$, $N_{min} = 7$	**M1** **M1** **A1**	3	**8**
7	Find $F(x)$ for $1 \leqslant x \leqslant 4$:	$F(x) = (x^3 - 1)/63$	**B1**		
	Find $G(y)$ from $Y = X^2$ for $1 \leqslant x \leqslant 4$: (result may be stated)	$G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63$	**M1 A1**		
	Find $g(y)$ for corresponding range of y:	$g(y) = y^{1/2}/42$ **A.G.**	**A1**		
	Find or state corresponding range of y :	$1 \leqslant y \leqslant 16$ **A.G.**	**B1**	5	
(i)	Find median value m of Y:	$(m^{3/2} - 1)/63 = ½$ $m = 32·5^{2/3} = 10·2$	**M1 A1**	2	
(ii)	Find $E(Y)$ [or equivalently $E(X^2)$]:	$E(Y) = \int y\, g(y)\, dy = \int y^{3/2}\, dy /42$ $= [y^{5/2}]_1^{16}/105 = 1023/105$ $= 341/35$ *or* $9·74$	**M1 A1**	2	**9**

Question Number	Mark Scheme Details			Part Mark	Total
8	Find mean of sample data [for use in Poisson distn.]:	$\lambda = 220/100 = 2{\cdot}2$	**B1**		
	State (at least) null hypothesis (AEF):	H_0: Poisson distn. fits data *or* $\lambda = 2{\cdot}2$	**B1**		
	Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.): (ignore incorrect final value here for M1)	11·080 24·377 26·814 19·664 10·8151 4·759 2·491	**M1 A1**		
	Combine last two cells so that exp. value ⩾ 5:	O_i: 3 E_i: 7·25	**M1***		
	Calculate value of χ^2 (to 2 d.p.; A1 dep M1*): (allow 7·95 if 1 d.p. exp.values used)	$\chi^2 = 0{\cdot}076 + 2{\cdot}879 + 0{\cdot}653 + 1{\cdot}448 + 0{\cdot}441 + 2{\cdot}491$ $= 7{\cdot}99$	**M1 A1**		
	State or use consistent tabular value (to 3 s.f.):	5 cells: $\chi_{3,0.95}{}^2 = 7{\cdot}815$ 6 cells: $\chi_{4,0.95}{}^2 = 9{\cdot}488$ (correct) 7 cells: $\chi_{5,0.95}{}^2 = 11{\cdot}07$	**B1**		
	State or imply valid method for conclusion e.g.:	Accept H_0 if χ^2 < tabular value	**M1**		
	Conclusion (AEF, requires both values correct): Not combining cells [so $\chi^2 = 8{\cdot}64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7)	Distn fits *or* $\lambda = 2{\cdot}2$	**A1**	10	**10**
9	Calculate gradient b_1 in $y - \bar{y} = b_1(x - \bar{x})$:	$S_{xy} = 24\,879 - 472 \times 400/8$ $= 1\,279$ $S_{xx} = 29\,950 - 472^2/8 = 2\,102$ $b_1 = S_{xy} / S_{xx} = 0{\cdot}6085$ (3 s.f.)	**M1 A1**		
	Find regression line of y on x:	$y = 400/8 + b_1 (x - 472/8)$ $= 50 + 0{\cdot}6085 (x - 59)$ $= 0{\cdot}6085x + 14{\cdot}1$	**M1 A1**		
	Find y when $x = 72$: Allow use of regression line of x on y (since neither variable clearly independent):	$= 57{\cdot}9$ *or* 58 $S_{yy} = 21\,226 - 400^2/8 = 1\,226$ $b_2 = S_{xy} / S_{yy} = 1{\cdot}043$ **(M1 A1)** $x = 472/8 + b_2 (y - 400/8)$ **(M1 A1)** $= 1{\cdot}043\,y + 6{\cdot}85$ $Y = 62{\cdot}5$ *or* 62	**(A1)**	5	

Question Number	Mark Scheme Details			Part Mark	Total
	Find product moment correlation coefficient r:	$r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ or $\sqrt{(0{\cdot}6085 \times 1{\cdot}043)} = 0{\cdot}797$	**M1 A1***		
	State both hypotheses (B0 for r …):	H_0: $\rho = 0$, H_1: $\rho \neq 0$	**B1**		
	State or use correct tabular two-tail r-value:	$r_{8,\,5\%} = 0{\cdot}707$	**B1***		
	State or imply valid method for conclusion e.g.:	Reject H_0 if $\lvert r\rvert$ > tab. value (AEF)	**M1**		
	Correct conclusion (AEF, dep A1*, B1*):	There is non-zero correlation	**A1**	6	**11**
10A	Find MI of lamina about Q:	$I_{\text{lamina}} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\} + m(9a/2)^2$ [$= (15/4 + 81/4)\,ma^2 = 24\,ma^2$]	**M1 A1**		
	State or find MI of rod about Q:	$I_{\text{rod}} = (\frac{1}{3} + 1)\,M\,(3a/2)^2$ [$= 3Ma^2$]	**B1**		
	Sum to find MI of object about Q:	$I_1 = 24\,ma^2 + 3\,Ma^2 = 3\,(8m + M)\,a^2$ **A.G.**	**A1**	4	
	Find MI of object about mid-point of PQ:	$I_2 = (15/4 + 3^2)\,ma^2 + \frac{1}{3}\,M\,(3a/2)^2 = (51/4)\,ma^2 + \frac{3}{4}\,Ma^2 = \frac{3}{4}\,(17m + M)\,a^2$ **A.G.**	**M1 A1**	2	
	Use eqn of circular motion to find $\mathrm{d}^2\theta/\mathrm{d}t^2$ for axis l_1:	$[-]I_1\,\mathrm{d}^2\theta/\mathrm{d}t^2 = mg \times (9a/2)\sin\theta + Mg \times (3a/2)\sin\theta$ [$= (9m/2 + 3M/2)\,ga\sin\theta$]	**M1 A1**		
	[Approximate $\sin\theta$ by θ and] find ω_1^2 in SHM eqn:	$\omega_1^2 = (3m + M)g / 2(8m + M)\,a$	**M1**		
	Find period T_1 for axis l_1 from $2\pi/\omega_1$: (AEF)	$T_1 = 2\pi\sqrt{\{2(8m + M)\,a / (3m + M)g\}}$	**A1**		
	Use eqn of circular motion to find $\mathrm{d}^2\theta/\mathrm{d}t^2$ for axis l_2:	$[-]I_2\,\mathrm{d}^2\theta/\mathrm{d}t^2 = mg \times 3a\sin\theta$	**M1**		
	[Approximate $\sin\theta$ by θ and] find ω_2^2 in SHM eqn:	$\omega_2^2 = 4mg / (17m + M)\,a$	**M1**		
	Find period T_2 for axis l_2 from $2\pi/\omega_2$: (AEF)	$T_2 = 2\pi\sqrt{\{(17m + M)\,a / 4mg\}}$	**A1**		
	Verify that $T_1 = T_2$ when $m = M$: (AEF) [Taking $m = M$ throughout 2nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]	$T_1 = 2\pi\sqrt{(18\,a / 4g)} = T_2$	**B1**	8	**14**

Question Number	Mark Scheme Details			Part Mark	Total
10B	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = \mu_Y$, H_1: $\mu_X \neq \mu_Y$	**B1**		
	State assumption (AEF):	Distributions have equal variances	**B1**		
	Find sample means and estimate popln. variances: (allow biased here: $0{\cdot}36$ *or* $0{\cdot}6^2$) (allow biased here: $0{\cdot}4183$ *or* $0{\cdot}6468^2$)	$\bar{x} = 4{\cdot}2$, $\bar{y} = 4{\cdot}8$ $s_X^2 = (180 - 42^2/10) / 9$ $= 0{\cdot}4$ *or* $0{\cdot}6325^2$ $s_Y^2 = (281{\cdot}5 - 57{\cdot}6^2/12) / 11$ $= 0{\cdot}4564$ *or* $251/550$ *or* $0{\cdot}6755^2$	**M1**		
	Estimate (pooled) common variance: (note s_X^2 and s_Y^2 not needed explicitly)	$s^2 = (9\,s_X^2 + 11\,s_Y^2) / 20$ (AEF) *or* $(180 - 42^2/10 + 281{\cdot}5 - 57{\cdot}6^2/12) / 20$ $= 0{\cdot}431$ *or* $0{\cdot}6565^2$	**M1 A1**		
	Calculate value of t (to 3 s.f.):	[-] $t = (\bar{y} - \bar{x}) / s\,\sqrt{(1/10 + 1/12)}$ $= 2{\cdot}13$	**M1 A1**		
	State or use correct tabular t value: (or can compare $\bar{y} - \bar{x} = 0{\cdot}6$ with $0{\cdot}586$)	$t_{20,\,0.975} = 2{\cdot}086$ [allow $2{\cdot}09$]	**B1***		
	Correct conclusion (AEF, √ on t, dep *B1): **S.R.** Implicitly taking s_X^2, s_Y^2 as popln. variances: (may also earn first B1 B1 M1) Comparison with $z_{0.975}$ and conclusion: (can earn at most 5/9)	$t > 2{\cdot}09$ so mean masses not same $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ $= 0{\cdot}6 / \sqrt{(0{\cdot}078} = 2{\cdot}15$ $2{\cdot}15 > 1{\cdot}96$ so mean masses not same	**B1√** **(B1)**	9	
	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = 3{\cdot}8$, H_1: $\mu_X > 3{\cdot}8$ *or* H_0: $\mu_X = \mu_Z$, H_1: $\mu_X > \mu_Z$	**B1**		
	Calculate value of t using s_X from above:	$t = (4{\cdot}2 - 3{\cdot}8) / (s_X/\sqrt{10}) = 2{\cdot}0$	**M1 A1**		
	State or use correct tabular t value: (or can compare $0{\cdot}4$ with $0{\cdot}367$)	$t_{9,\,0.95} = 1{\cdot}833$ [allow $1{\cdot}83$]	**B1***		
	Correct conclusion (A.E.F., √ on t, dep *B1):	$t > 1{\cdot}833$, so claim is justified *or* mean mass of Royals > mean mass of Crowns	**B1√**	5	**14**