Cambridge Assessment International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/02**

Paper 2 Further Pure Mathematics 2 **For examination from 2020**

MARK SCHEME

Maximum Mark: 75

Specimen

This document has **14** pages. Blank pages are indicated.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:

Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:

Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:

Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mark Scheme Notes

Marks are of the following three types.

M	Method mark, given for a valid method applied to the problem. Method marks can still be given even if there are numerical errors, algebraic slips or errors in units. However the method must be applied to the specific problem, e.g. by substituting the relevant quantities into a formula. Correct use of a formula without the formula being quoted earns the M mark and in some cases an M mark can be implied from a correct answer.
A	Accuracy mark, given for an accurate answer or accurate intermediate step following a correct method. Accuracy marks cannot be given unless the relevant method mark has also been given.
B	Mark for a correct statement or step.
DM or DB	M marks and B marks are generally independent of each other. The notation DM or DB means a particular M or B mark is dependent on an earlier M or B mark (indicated by *). When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT below).
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures (sf) or would be correct to 3 sf if rounded (1 decimal place (dp) in the case of an angle in degrees). As stated above, an A or B mark is not given if a correct numerical answer is obtained from incorrect working.
- Common alternative solutions are shown in the Answer column as: '**EITHER** Solution 1 **OR** Solution 2 **OR** Solution 3 …'. Round brackets appear in the Partial Marks column around the marks for each alternative solution.
- The total number of marks available for each question is shown at the bottom of the Marks column in bold type.
- Square brackets [ ] around text show extra information not needed for the mark to be awarded.

The following abbreviations may be used in a mark scheme.

AG	Answer given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid).
CAO	Correct answer only (emphasising that no 'follow through' from an error is allowed).
CWO	Correct working only
FT	Follow through after error (see Mark Scheme Notes for further details).
ISW	Ignore subsequent working
OE	Or equivalent form
SC	Special case
SOI	Seen or implied

Question	Answer	Marks	Partial Marks	Guidance
1	$m^2 + 4m + 4 = 0$: $m = –2, –2$	1	**M1**	Auxiliary equation
	$x = (At + B)e^{-2t}$	1	**A1**	Complementary function
	$x = P + Qt + rt^2$: $\dot{x} = Q + 2Rt$: $\ddot{x} = 2R$	1	**B1**	Particular integral and its derivatives
	Substitute and equate coefficients of powers of t	1	**M1**	
	$P = 1$, $Q = 1$, $R = -\frac{1}{2}$	1	**A1**	
	$x = (At + B)\ e^{-2t} + 1 + t - \frac{1}{2}t^2$	1	**A1**	
		6		

Question	Answer	Marks	Partial Marks	Guidance
2	$3 + 4x - 4x^2 = 4 - (2x - 1)^2$	1	**M1**	Complete the square
	EITHER Solution 1 $2x - 1 = 2\sin\theta$: $\sqrt{4 - 4\sin^2\theta} = 2\cos\theta$:	1	**(M1**	Use appropriate substitution
	Integral = $\int \frac{\cos\theta}{2\cos\theta} \mathrm{d}\theta = \frac{1}{2}\theta = \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{2}\right)$	2	**M1A1)**	Integrate
	OR Solution 2 $\frac{1}{2}\int \frac{1}{\sqrt{1 - \left(x - \frac{1}{2}\right)^2}} \mathrm{d}x$	1	**(M1**	Use formula
	$= \frac{1}{2}\sin^{-1}\left(x - \frac{1}{2}\right)$	2	**M1A1)**	
	$\frac{1}{2}\left(\sin^{-1}(0.5) - \sin^{-1}(-0.5)\right)$	1	**M1**	Use limits
	$= \frac{\pi}{6}$	1	**A1**	CAO NOTE: An answer of $\frac{\pi}{6}$ without working scores 0 (as per front cover instructions)
	Available marks	**6**		

Question	Answer	Marks	Partial Marks	Guidance
3	$\frac{\mathrm{d}y}{\mathrm{d}x}+3\frac{y}{x}=\frac{\sin x}{x^2}$	1	**B1**	Divide through by x
	Integrating factor: $\mathrm{e}^{\int \frac{3}{x}\mathrm{d}x}$	1	**M1**	Find integrating factor
	$=x^3$	1	**A1**	
	$\frac{\mathrm{d}}{\mathrm{d}x}(yx^3)=x\sin x$	1	**M1**	Correct form LHS and attempt to integrate RHS
	$yx^3=-x\cos x+\sin x+c$	1	**A1**	
	$c=-1$	1	**M1**	Find c
	$y=\frac{-x\cos x+\sin x-1}{x^3}$	2	**M1A1**	Division through by coefficient of y. CAO
		8		

Question	Answer	Marks	Partial Marks	Guidance
4(a)	Sum = $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$	2	**M1A1**	Heights of rectangles correct
	$<\int_1^n \frac{1}{x^2}\mathrm{d}x$	1	**M1**	Compare with integral
	$=1-\frac{1}{n}$	1	**M1**	Evaluate integral
	$\sum_{r=1}^{n}\frac{1}{r^2}<2-\frac{1}{n}=\frac{2n-1}{n}$	1	**A1**	AG Deal with the 1 and obtain given answer
		5		
4(b)	**EITHER** Solution 1 Sum area of appropriate set of rectangles =	1	**(M1**	Appropriate rectangles
	$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}>\int_1^{n+1}\frac{1}{x^2}\mathrm{d}x$	1	**M1**	Complete method, obtaining inequality involving sum of rectangle areas and appropriate integral
	So $\sum_{r=1}^{n}\frac{1}{r^2}>1-\frac{1}{n+1}$	1	**A1)**	
	OR Solution 2 Sum area of appropriate set of rectangles =	1	**(M1**	Appropriate rectangles
	$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}>\int_1^{n}\frac{1}{x^2}\mathrm{d}x=1-\frac{1}{n}$	1	**M1**	Complete method, obtaining inequality involving sum of rectangle areas and appropriate integral
	So $\sum_{r=1}^{n}\frac{1}{r^2}>1-\frac{1}{n}+\frac{1}{n^2}\left(=\frac{n^2-n+1}{n^2}\right)$	1	**A1)**	
	Available marks	**3**		

Question	Answer	Marks	Partial Marks	Guidance
5(a)	$\dot{x} = \mathrm{e}^t - 4;\ \dot{y} = 4\mathrm{e}^{\frac{1}{2}t}$	1	**B1**	find dx/dt and dy/dt
	$\mathrm{d}s = \sqrt{\mathrm{e}^{2t} + 16 + 8\mathrm{e}^t} = \mathrm{e}^t + 4$	2	**M1A1**	
	Arc length $= \int_0^2 (\mathrm{e}^t + 4)\,\mathrm{d}t = [\mathrm{e}^t + 4t]_0^2 = \mathrm{e}^2 + 7$	2	**M1A1**	
		5		
5(b)	$S = 2\pi \int_0^2 8\mathrm{e}^{\frac{1}{2}t}(\mathrm{e}^t + 4)\,\mathrm{d}t$	1	**B1FT**	FT
	$= 16\pi \int_0^2 (\mathrm{e}^{1.5t} + 4\mathrm{e}^{0.5t})\,\mathrm{d}t$	1	**M1**	limits not required
	$16\pi\left[\frac{2}{3}\mathrm{e}^{1.5t} + 8\mathrm{e}^{0.5t}\right]_0^2$	2	**M1A1**	correct integration
	$= 16\pi\left(\frac{2}{3}\mathrm{e}^3 + 8\mathrm{e} - \frac{26}{3}\right)$	1	**A1**	CAO
		5		

Question	Answer	Marks	Partial Marks	Guidance
6(a)	$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$	1	**B1**	
	$= c^5 + 5\mathrm{i}c^4s - 10c^3s^2 - 10\mathrm{i}c^2s^3 + 5cs^4 + \mathrm{i}s^5$	2	**M1A1**	Binomial expansion
	$\tan 5\theta = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$	1	**M1**	
	$\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$	1	**A1**	AG Division of each term by c^5 clearly stated
		5		
6(b)	Roots of $\tan 5\theta = 0$ are $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$	1	**B1**	Allow inclusion of 0 here
	$t^4 - 10t^2 + 5 = 0$ has roots $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$	1	**B1**	
	$\left(t - \tan\frac{\pi}{5}\right)\left(t - \tan\frac{2\pi}{5}\right)\left(t - \tan\frac{3\pi}{5}\right)\left(t - \tan\frac{4\pi}{5}\right) = 0$	1	**M1**	
	Since $\tan\frac{4\pi}{5} = -\tan\frac{\pi}{5}$ and $\tan\frac{3\pi}{5} = -\tan\frac{2\pi}{5}$, $\left(t^2 - \tan^2\left(\frac{\pi}{5}\right)\right)\left(t^2 - \tan^2\left(\frac{2\pi}{5}\right)\right) = 0$	1	**M1**	
	So roots of $x^2 - 10x + 5 = 0$ are $\tan^2\left(\frac{\pi}{5}\right)$ and $\tan^2\left(\frac{2\pi}{5}\right)$	1	**A1**	AG
		5		

Question	Answer	Marks	Partial Marks	Guidance
7(a)	$x = \dfrac{e^{u} - e^{-u}}{e^{u} + e^{-u}}$	1	**M1**	Write in exponential form
	$e^{2u}(1 - x) = 1 + x$	1	**M1**	Rearrange
	$u = \tanh^{-1} x = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	1	**A1**	AG CAO
		3		
7(b)	**EITHER** Solution 1 $y = \frac{1}{2}\ln\left(\dfrac{1 + \dfrac{1-x}{2+x}}{1 - \dfrac{1-x}{2+x}}\right)$	1	**(M1**	Use result of (a)
	$= \frac{1}{2}\ln\left(\dfrac{3}{2x+1}\right)$	1	**A1**	
	Differentiate: $y' = \frac{1}{2} \times \dfrac{2x+1}{3} \times \dfrac{-6}{(2x+1)^2}$	1	**M1**	Any equivalent work, e.g. splitting the log into 2 terms
	$= \dfrac{-1}{2x+1}$ giving $(2x+1)\dfrac{dy}{dx} + 1 = 0$	1	**A1)**	AG
	OR Solution 2 Differentiate first: $\operatorname{sech}^2 y \dfrac{dy}{dx} = \dfrac{-3}{(2+x)^2}$	2	**(M1A1**	
	Use $\operatorname{sech}^2 y = 1 - \tanh^2 y$	1	**M1**	
	to give $(2x+1)\dfrac{dy}{dx} + 1 = 0$	1	**A1)**	AG
	Available marks	**4**		

Question	Answer	Marks	Partial Marks	Guidance
7(c)	Differentiate again: $y'' = \dfrac{2}{(2x+1)^2}$	1	**B1**	Or differentiate result in (b): $(2x+1)y'' + 2y' = 0$
	$y(0) = \tanh^{-1}\left(\dfrac{1}{2}\right)$	1	**M1**	Values of derivatives at 0
	$y'(0) = -1;\ y''(0) = 2$	1	**M1**	Use Maclaurin's series
	$y = \frac{1}{2}\ln 3 - x + 2 \times \dfrac{x^2}{2}$	1	**M1**	In terms of ln
	$= \frac{1}{2}\ln 3 - x + x^2$	1	**A1**	CAO
		5		

Question	Answer	Marks	Partial Marks	Guidance
8(a)(i)	$\begin{vmatrix} 1 & -2 & -2 \\ 2 & a-9 & -10 \\ 3 & -6 & 2a \end{vmatrix} = 0$ $2a(a-9) - 60 + 2(4a+30) - 2(-12-3a+27) = 0$	2	**M1A1**	Write and evaluate determinant
	$2a^2 - 4a - 30 = 0$ $a = 5$ or -3	1	**M1**	
	Unique solution for $a \neq 5$, $a \neq -3$	1	**A1**	
		4		
8(a)(ii)	1st and 3rd equations: $x - 2y - 2z + 7 = 0$ $x - 2y - 2z + \frac{29}{3} = 0$ Inconsistent, so no solution.	1	**B1**	
	These two equations represent parallel planes. Other equation represents a non-parallel plane which intersects each of the other two in a line.	2	**B1B1**	
		3		

Question	Answer	Marks	Partial Marks	Guidance
8(b)(i)	$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$	1	**B1**	Equate determinant to zero
	$(1-\lambda)(\lambda-2)(\lambda-3) = 0$	1	**M1**	Expand determinant and factorise
	$\lambda = 1, 2, 3$	2	**A1A1**	Award A1A0 for 2 correct solutions, A0A0 for 1 correct solution.
		4		
8(b)(ii)	$\mathbf{A}$ satisfies its characteristic equation, so $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - 6\mathbf{I} = \mathbf{0}$	1	**B1**	
	Multiply through by $\mathbf{A}^{-1}$ to give $6\mathbf{A}^{-1} = \mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I}$	1	**M1**	For information: $\mathbf{A}^2 = \begin{pmatrix} -1 & 5 & 10 \\ -2 & 6 & 10 \\ -4 & 4 & 9 \end{pmatrix}$
	$\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 \\ -2 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$	2	**M1A1**	
		4		

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